The Connection Between Longitudinal and Lateral Web Dynamics

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IWEB 2019

The elusive entry angle

- Where does the entry angle come from? This paper will show that the answer to this question reveals a connection between longitudinal and lateral behavior that has gone largely unnoticed.

The elusive entry angle

- In beam models, entry angle refers to the angle between the tangent to the web centerline and the normal to the roller axis at the line of entry onto the roller.
- Whenever the entry angle becomes non-zero, a web that is moving longitudinally through a process will also move laterally on the roller in a direction that returns the entry angle to zero.

The elusive entry angle

- If the web is modeled as a perfectly flexible string, this behavior is intuitively obvious because it bends sharply on entering a roller that is pivoted or shifted laterally.
- However, in the case of the most commonly used Euler-Bernoulli (E-B) beam model, the web can't make a sharp bend. If it is initially perpendicular to the roller axis, beam theory says that, provided there is no slipping, it should remain perpendicular as the roller is shifted or pivoted and thus wouldn't move.
- We know from experience, however, that a real moving web begins to move laterally soon after a roller pivots or shifts? So, how can this be?

Normal entry equation

- The entry angle enters into lateral dynamic analysis through this equation. It is commonly referred to as the normal entry equation.

Lateral web velocity

Slope

Roller angle

$$\frac{dy_L}{dt} = V\left(\theta_r - \frac{dy_L}{dx}\right) + \frac{dz}{dt}$$

Entry angle

MD web speed

Roller velocity

A naïve lateral dynamic solution

- Assume for a moment that you’re a new web handling researcher who has never done an experiment on real web.
- You might develop a model in the following way.

Shape of the web

- Slope is calculated from a static analysis of web shape. This is the equation for lateral position $y(x)$ that results when face angles ϕ_L, ϕ_0 and lateral displacements, y_L, y_0 are chosen as boundary conditions.
- The shape factors, g_4, g_5 and g_6 depend on span dimensions, mechanical parameters of the web and distance along the span.

$$y(x) = y_0 + \left(y_0 - y_L\right) g_4(x, K, L) + \phi_L g_5(x, K, L) + \phi_0 g_6(x, K, L)$$

- Derivation of this equation is documented in Appendix A of the paper.

Slope

- Taking the derivative of the previous equation and substituting L for x we get the slope at the line of entry to the downstream roller.

$$\frac{dy_L}{dx} = \left(y_0 - y_L\right)\frac{h_1(K,L)}{L} + \phi_L h_2(K,L) + \phi_0 h_3(K,L)$$

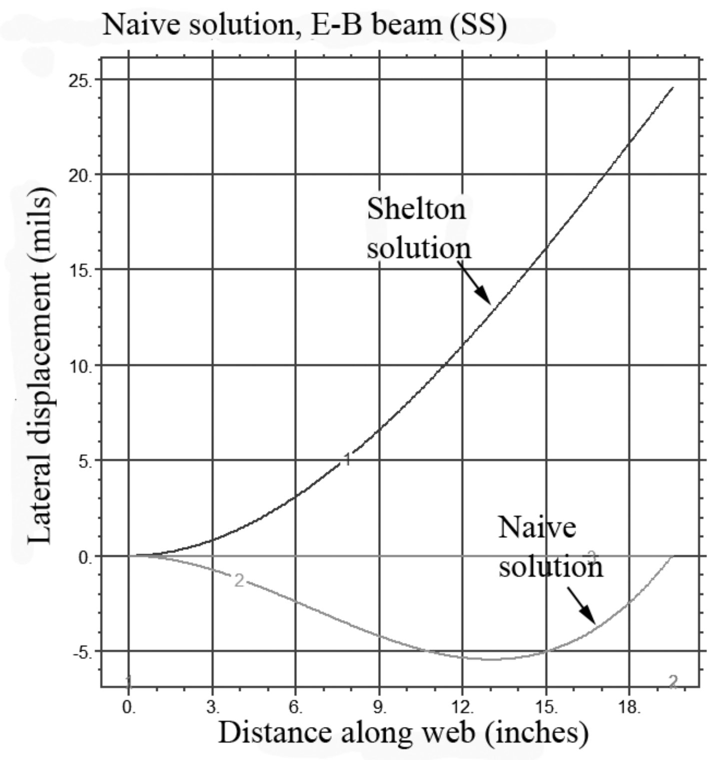
A naïve solution for y_L

- The face angle ϕ_L in the previous equation is set equal to the roller angle θ_r. Making this change and substituting into the normal entry equation, the following relationship is obtained.

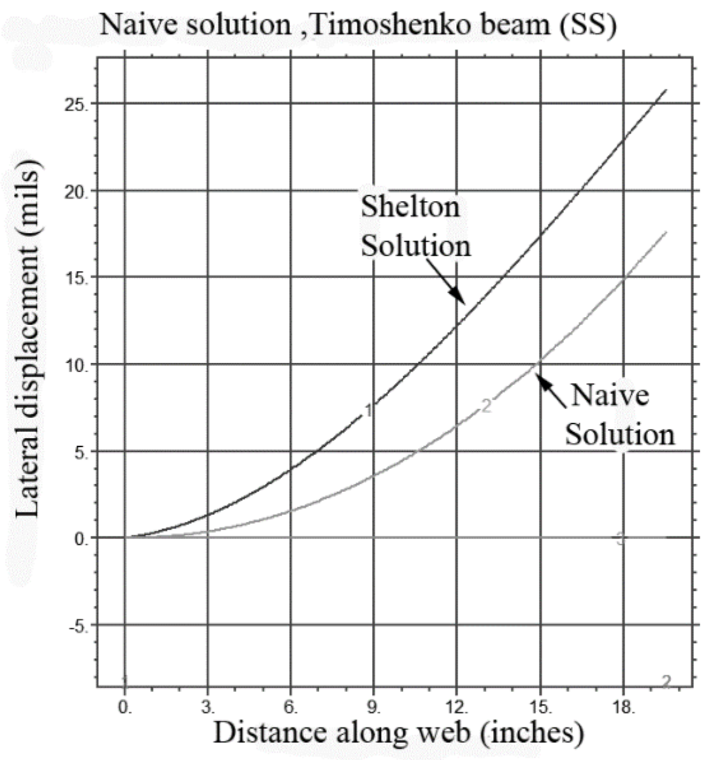
$$\frac{dy_L}{dt} = V_o\left[\theta_r - \left(y_0 - y_L\right)\frac{h_1(K,L)}{L} - \theta_r h_2(K,L) - \phi_0 h_3(K,L)\right] + \frac{dz_L}{dt}$$

- This can be solved for y_L To keep things simple, assume that y_0 and ϕ_0 are zero - upstream span in a state of uniform uniaxial stress.

Solutions after 5 time constants



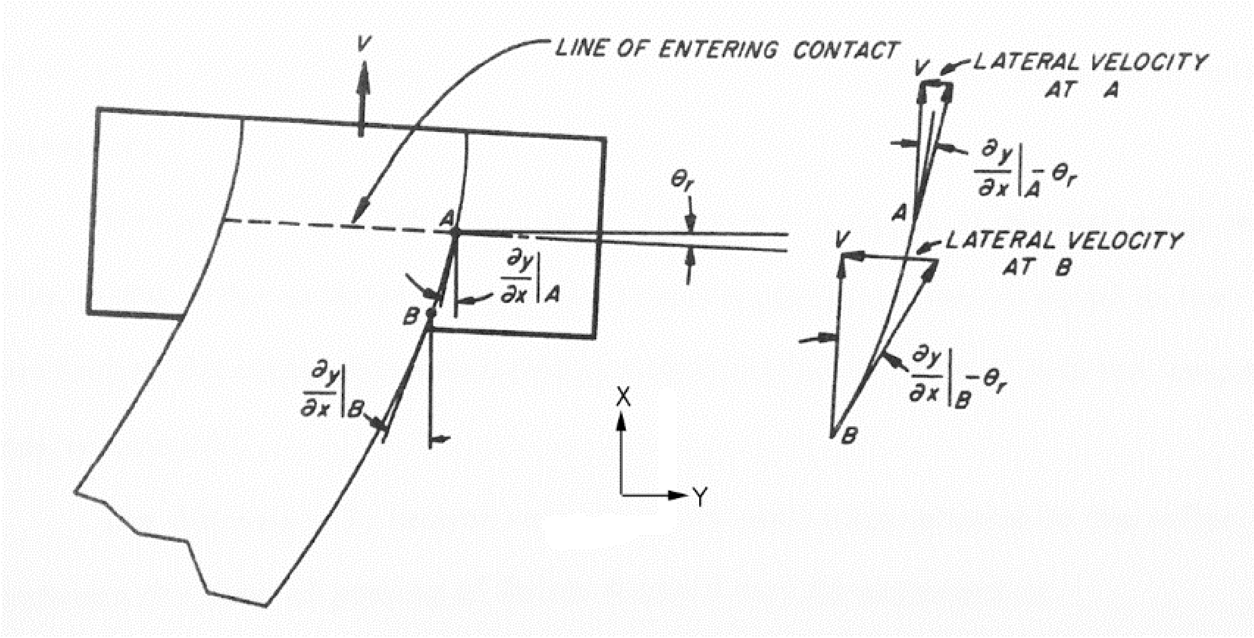
Entry angle stays fixed at zero.



Entry angle changed, but position falls short.

Shelton’s solution

- Shelton, in his seminal dissertation on lateral web dynamics, argued that the entry angle changes because slope variation due to upstream web curvature is transported onto the roller by the web’s longitudinal motion.



Acceleration equation

- Using this idea led to an equation for acceleration.
- This is used as one of two dynamic boundary conditions (the normal entry equation is the other) in a second order solution based on the shape equation for curvature.

$$\underbrace{\frac{d^2 y_L}{dt^2}}_{\text{Web acceleration}} = \underbrace{V^2}_{\text{Web speed}} \underbrace{\frac{d^2 y_L}{dx^2}}_{\text{Curvature}} + \underbrace{\frac{d^2 z}{dt^2}}_{\text{Roller acceleration}}$$

Problem with acceleration equation

- The problem with curvature transport is that it produces an expression for lateral acceleration that conflicts with the definition obtained by simply differentiating the normal entry equation. Shelton was aware of this and made the following comment about it.
- *"Note that Equation 4.1.5 [the acceleration equation] is not merely the derivative of Equation 4.1.2 [the normal entry equation]; differentiation of the latter equation results in an extra term containing the velocity of roller swivelling, dθ/dt. Because of the assumption that shear deflection is negligible, no acceleration can occur as an instantaneous result of roller swiveling. But only indirectly as the web curvature changes. A suddenly swivelling roller instantaneously swivels the downstream end of the web an equal amount, so that no instantaneous change in steering rate occurs, in contrast to the first-order theory of Chapter III [which employs a flexible string model of the web]."*

But it works

- Still, at any moment there can be only one value for lateral acceleration and if the normal entry equation is valid, there is no reason to think that its time derivative wouldn’t provide it.
- However, regardless of any concern about it, Shelton showed in his dissertation that using the acceleration equation in a dynamic model produced excellent agreement with experiments.
- He tested four configurations; a parallel pair with KL = 2, a parallel pair with KL =10, an oversteering guide and an understeering guide. Amplitude and phase response were measured in each case at six different frequencies.

Benson's method

- Benson, in a 2002 paper, found a better way to derive the acceleration equation. He started by assuming that the pivoting velocities of the roller angle, θ_r, and web face angle, ϕ_L, must match at the line of entry of the web onto the roller. He then applied the material derivative and arrived at the following expression.

$$\frac{d\theta_r}{dt} = \frac{D\phi_L}{Dt} = \frac{d\phi_L}{dt} + V\frac{d\phi_L}{dx}$$

Benson’s method

- Benson chose to organize his model in the form of four first-order equations. So, he wasn’t interested in anything like an acceleration equation as a boundary condition. Nevertheless, to help establish the validity of his model, he showed that his material derivative equation could be used to derive it.
- The result of his derivation, shown in the next slide, included the effect of shear and is in agreement with the expression derived by myself from considerations of mass transfer.

Benson's method

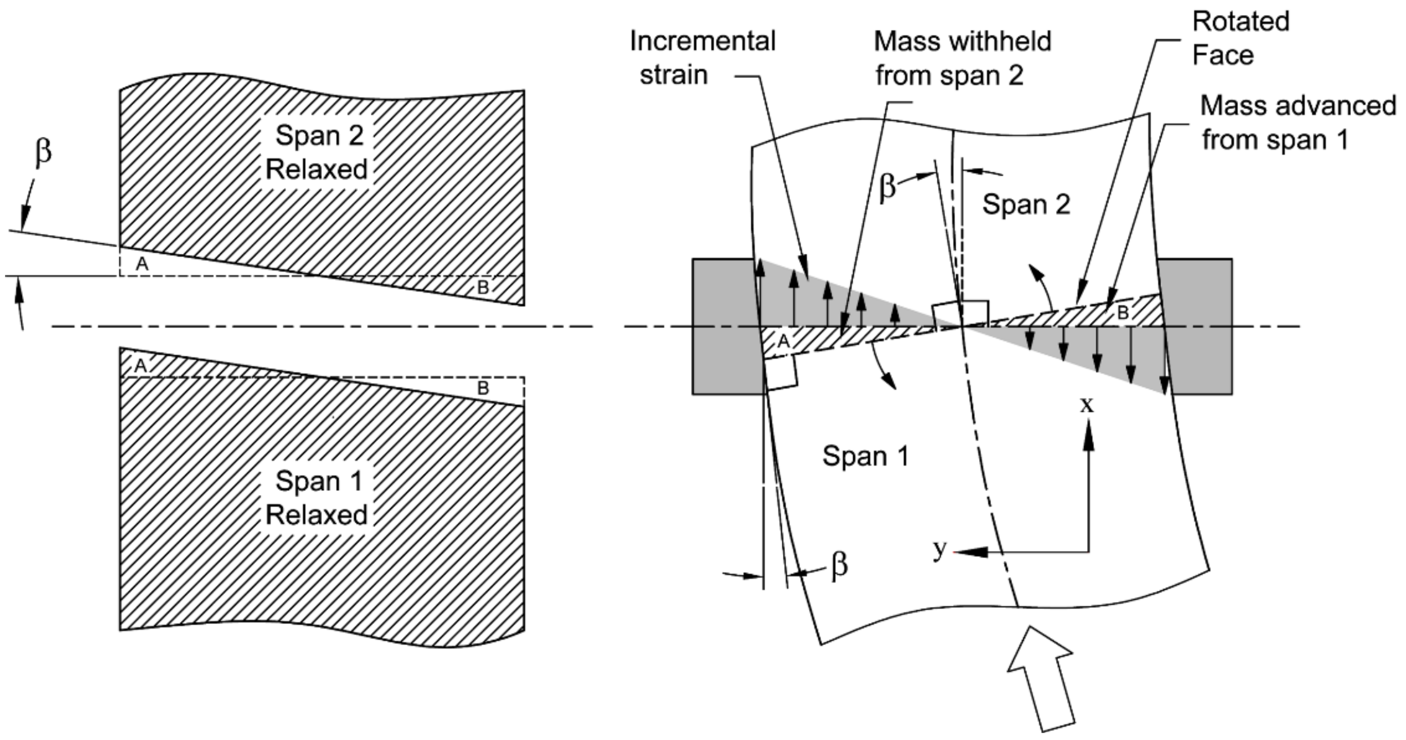
$$\frac{d^2 y_L}{dt^2} = V^2 \frac{d^2 y_L}{dx^2} + \frac{d^2 z_L}{dt^2} - V\left(\frac{d\psi_L}{dt} + V\frac{d\psi_L}{dx}\right)$$

- When shear ψ_L is eliminated, this defaults to Shelton's acceleration equation – as it should.

The entry angle is caused by longitudinal transfer of mass

- In a 2017 IWEB paper I showed that the entry angle of the normal entry equation is entirely due to the effect of mass transferred longitudinally between spans. I won’t repeat the details of the analysis of that paper, but will, instead, summarize the principle results.

Effect of mass transfer after roller shift



Effect of mass transfer after roller shift

- The net effect of the mass transfer is angular rotation of the face of the web at the line of entry through an angle, β. There is no slipping involved in the formation of β. It is entirely due to variations in mass flow that change the relationship between face angle and roller angle from $\phi_L = \theta_r$ to $\phi_L = \theta_r + \beta$. Analysis of the relationship between the strain profile and mass flow shows that in the absence of shear deformation, β is defined as,

$$-\frac{d\beta}{dt} = \frac{d}{dt}\left(\theta_r - \frac{dy_L}{dx}\right) = V\frac{d^2 y_L}{dx^2}$$

β with shear deformation

$$-\frac{d}{dt}\left(\beta+\psi_L\right)=\frac{d}{dt}\left(\theta_r-\frac{dy_L}{dx}\right)=aV\frac{d^2y_L}{dx^2}-\frac{d\psi}{dt}$$

$$a=1+\frac{nT}{AG}$$

Why worry about mass transfer?

- It is fair to ask why mass transfer is needed. After all, Shelton’s and Benson’s equations aren’t wrong. The answer is that it is an essential part of the physical picture that has been missing and, as will be shown in a companion paper, it is the key to understanding how to combine lateral and longitudinal behavior in a single model.

More connections with the methods of Shelton and Benson

- Equating the two values of acceleration from Shelton's model (the acceleration equation and the time derivative of velocity from the normal entry equation) produces a key relationship from mass transfer analysis.

$$\frac{d}{dt}\left(\theta_r - \frac{dy_L}{dx}\right) = V\,\frac{d^2 y_L}{dx^2}$$

- Substituting $dy_L/dx = \phi_L$ (true for an E-B beam) in Benson's material derivative also produces this equation.

Why did Benson’s velocity matching work?

- In its relaxed state, all the particles in a uniform web are assumed to be moving in straight lines aligned with the x-axis. As the web deforms, those paths become streamlines that are congruent with the web shape. It is these curved streamlines to which the shape equations apply.
- It is important to realize, however, that in a moving web, the particles following those paths will not all be travelling at the same speed. For example, particles on the outside edge of a curve, and not in proximity to a roller, will be travelling faster than particles on the inside edge. Then, when they arrive at the roller, where, it is assumed, they will “stick” to its surface, they must take on its velocity. That velocity is the same at all points along the line of entry.

Why did Benson’s velocity matching work?

- Benson recognized this fact in his velocity matching boundary condition when he said, “It is further expected that the web will stick to the roller for all points of first contact – not just at the web’s centerline. To achieve that, we must also match the rotational velocities of the roller and the web.”
- That can only happen if the rate of mass flow changes. So, although he made no mention of it, velocity matching at the roller effectively engages the mathematics of mass transfer.

Why did Benson’s velocity matching work?

- I owe Dilwyn Jones a debt of gratitude for reviewing early versions of this paper and patiently defending Benson’s method. I was inclined to distrust anything that didn’t explicitly mention mass conservation, but he convinced me of its validity, using an argument similar to the one I just made.

Conclusion

- Mass transfer, in the form of the continuity equation, has been part of tension analysis for decades, but it has not been used explicitly in the analysis of lateral behavior. It is now clear, however, that it is a vital part of the conceptual framework for both subjects.