PROPAGATION OF LONGITUDINAL TENSION IN A SLENDER MOVING WEB

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ABSTRACT

This paper presents a model for longitudinal tension propagation in a narrow web. There have been numerous investigations of transverse (out-of- plane) oscillations in a closely related application known as the "Traveling String". A few of these papers included consideration of the longitudinal oscillations which accompanied transverse oscillations. However, no attention has been given to longitudinal tension propagation as a principle feature of solid material transport.

The model is based on the one-dimensional wave equation, modified for a moving medium. Boundary conditions are developed that, for the first time, incorporate tension transfer and mass transport on rolling supports. A closed-form solution is developed using Laplace transforms.

A number of phenomena are described that will be of interest to process designers and troubleshooters. These can be used to explain existing tension problems, whose causes may have been unrecognized in the past, and to anticipate problems that will appear as line speeds are increased. Among these are:

1. Propagation of strain discontinuities when draw is increased suddenly.
2. Amplification of repetitive strain disturbances due to strain reflection and reinforcement.
3. Damping of solitary strain disturbances by transportation of energy out of the span.
4. Alteration of longitudinal resonant frequencies by transport motion.

Another important use of the model is to serve as a necessary step toward more advanced models that include out-of-plane motion, viscoelasticity and aerodynamics.

The model is tested by comparing it to the currently accepted O.D.E. model. At large time scales, where propagation phenomena are imperceptible, the two models are in good agreement.

NOMENCLATURE

A_o	cross sectional area of web in relaxed state
A1, A2, A3	cross sectional area of the web at the entry to roller A, exit of roller A and entry to roller B
C	longitudinal velocity of sound in the web material
E	Young's modulus of web
L	length of unsupported span between rollers A and B
s	laplace transform variable
t	time
T1, T2, T3	tension of the web at the entry to roller A, exit of roller A and entry to roller B
V1, V2, V3	velocity of the web at the entry to roller A, exit of roller A and entry to roller B
V_a	circumferential velocity of roller A
V_b	circumferential velocity of roller B
V_i	transport velocity of web
x	longitudinal position along web starting at roller A
σ_1, σ_2	density of web at the entry to roller A, and at the exit of roller A
ε_1	initial strain in web
σ_o	density of web in relaxed state
δv	amplitude of velocity change at roller B, m/s
ξ	displacement of a web particle from its rest position

INTRODUCTION

The subject of this paper is the behavior of longitudinal tension in a slender strip of flexible material, as it is transported in a continuous motion between support rollers.

Web process lines take many forms. But, the essential common element is that the material is a continuous flexible sheet (web) conveyed under tension over supports separated by open spans. A slender web is one in which the lateral dimensions are very small compared to the length of its open spans. For purposes of this analysis it can be considered a string. The supports are assumed to be rollers.

The model for this paper is based on the one-dimensional wave equation. Boundary conditions are developed which, for the first time, incorporate tension and mass transfer on rolling supports. The P.D.E. is solved analytically using Laplace transforms. The transform method is particularly convenient for investigating the response to a variety of forcing functions.

Solutions for step, single pulse, repetitive pulse and sine wave disturbances are presented along with a discussion of their implications for web processing.

BACKGROUND

Control of longitudinal speed and tension of webs is an important problem for web process designers. The tension must be high enough to ensure that the web remains free of wrinkles and conforms to the passline. It must be low enough to avoid permanent deformation and tearing. To date, most of the theoretical work on web tension has been directed at the problem of controlling the average tension. Very little attention has been given to the details of tension propagation within the span.

Current mathematical models for tension behavior [1] are based on equating the difference in mass flow rates entering and leaving a span to the time derivative of total mass in the span. This leads to a nonlinear ordinary differential equation relating average tension to the velocities at the entry and exit of a span. A linearized version has been developed for problems in which the velocities change by small amounts relative to the average line speed. An implicit assumption in deriving these equations is that tension can be assumed to be uniform throughout the span at all times. This is valid in any application being considered today because tension propagates very rapidly compared to web transport speed. For example, a web being processed at a typical speed of 500 ft/min would take 1.2 seconds to move through a 10 foot span. Assuming the web is made of polyester with a sound velocity of 3 x 105 ft/min, tension would propagate through this span in approximately .002 seconds. At 0.17% of the transport time, the propagation time is insignificant. This won't be the case forever though. Already, paper lines are running at speeds approaching 10,000 ft/min. The transport time for a span of 10 feet, at this speed, is 0.06 seconds.

TRAVELING STRING STUDIES

There have been numerous investigations of transverse oscillations in moving strings. The moving material in these studies is often described as a "Traveling String". Use of the word string is not only due to simplifying assumptions. Many of the authors were interested in the problems of transporting yarn from spools into weaving processes. The earliest publication was by Skutch [2] in 1897. Subsequent papers by Sack, 1954, [3], Archibald and Emslie 1958, [4], Swope and Ames, 1963, [5], Ames, Lee and Zaiser, 1968, [6], Ames and Vicario, 1969 [7], Kim and Tabarrock, 1972 [8], and Fox and Lilley, 1991, [9] dealt with additional features of transverse oscillations, such as damping, large amplitude nonlinearities, nonconservative energy changes, and computational methods. Thurman and Mote [10], 1969, presented an analysis of band saw blades that included flexural rigidity. Miranker, 1960, [11], who was motivated by problems with magnetic tape transport, was the first to observe that energy changes were nonconservative. Yang and Mote, 1991, [12] introduced a method for active control of transverse oscillations in a moving string using Laplace Transforms. A few of these papers included consideration of the longitudinal oscillations, which accompanied transverse oscillations - Ames, Lee and Zaiser; Ames and Vicario; Mote and Thurman. However, no attention has been given to longitudinal tension propagation as a principal feature of solid material transport.

THE PROBLEM

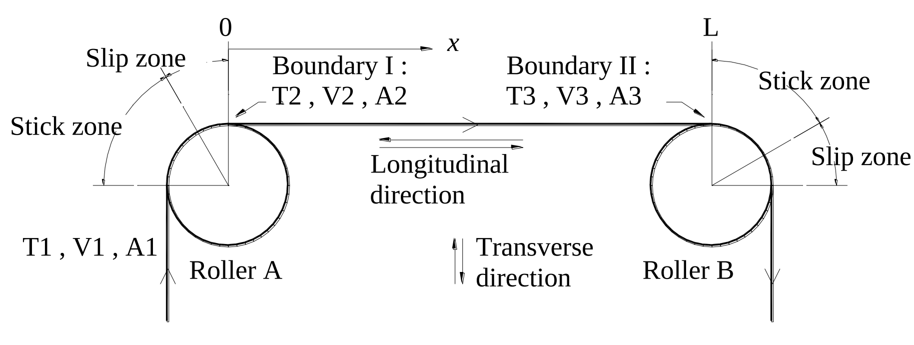

Figure 1 - Schematic of a Web Span

In the schematic of Fig. 1 it is assumed that:

1. Both rollers are driven and their speeds may be controlled accurately.
2. Coulomb friction exists between the web and the rollers.
3. The web obeys the familiar capstan relationship [13] while it is on the roller.
4. The web is uniform in its relaxed state.
5. The web is elastic in the longitudinal direction (obeys Hooke's law).
6. The web is perfectly flexible in the transverse direction.

Inputs to the problem are:

1. V_a = Circumferential velocity of roller A
2. V_b = Circumferential velocity of roller B
3. ε_1 = Strain at entry of roller A
4. A1 = Cross sectional area of web at entry or roller A
5. E = Young's modulus of the web

The model has two independent variables, x and t. Variable x is the position along the span. Variable t is time. The dependent variable is ξ. It is the displacement of web particles from their relaxed positions. The span has length L, starting at the exit of roller A and ending at the entry of roller B. In Fig. 1 the symbols T, V, and A refer, in the same order, to tension, velocity and cross sectional area.

There are a number of possible choices in setting up a particular problem. In a typical process line any one of three inputs, T1, V_a, or V_b could vary. However, in the next section it will become apparent that varying either of the first two variables leads to a nonlinear boundary condition requiring numerical methods for solution. Fortunately, most of the important features of tension propagation can be illustrated by varying V_b while holding T1 and V_a constant - a completely linear problem. This is the case that will be analyzed.

Although the principal topic of this paper is tension, most of the equations are formulated in terms of strain. Since Hooke's law is assumed, this creates no mathematical difficulties.

BOUNDARY CONDITIONS

Boundary conditions are needed that specify the particle velocities at the two ends of the span.

$$\frac{\partial \xi(0,t)}{\partial t} \quad and \quad \frac{\partial \xi(L,t)}{\partial t}$$

In statements like the one above, in which a constant appears as the argument of an operation, it will be understood that the substitution is made after the operation is performed. The expression on the left should be read as "the partial derivative of ξ as a function of t, evaluated at x = 0".

At the entry to roller B, the web will be in the stick zone, where coulomb friction and tension act to keep it from slipping. Therefore, at that boundary the speed will match the circumferential speed of the roller and the boundary condition is quite simple.

$$\frac{\partial \xi(L,t)}{\partial t} = V_b \qquad \text{Boundary Condition II} \qquad (1)$$

Boundary condition I is more complicated. Web particles exiting the slip zone of roller A won't match the roller speed. They change velocity as the web detaches from the roller surface and responds to the tension in the span. An exact analysis of conditions within the slip zone is complicated by the nonlinear effect of friction between the web and roller. No attempt will be made to do this. It is possible, however, to take account of the mass flow into and out of the zone using the principle of conservation of mass. Since the length of the slip zone is small compared to the total span, this should provide a reasonable approximation for Boundary I. So, at roller A:

$$A_1 \rho_1 V_1 = A_2 \rho_2 \frac{\partial \xi(0,t)}{\partial t} \qquad (2)$$

Web particles at the entry to roller A will be in a stick zone. So:

$$V_1 = V_a \qquad (3)$$

Expressions for A_1, ρ_1, A_2 *and* ρ_2 are determined as follows. Consider an increment of the web that in its relaxed state has a length l_0 cross sectional area, A_0 and density, ρ_0. When subjected to longitudinal stress, conservation of mass requires that the new values of area, A' and density, ρ' must conform to the following equation.

$$A' \rho' \, l_o (1+\varepsilon) = A_o \rho_o \, l_o \qquad (4)$$

The symbol ε is longitudinal strain. For infinitesimal lengths it is equivalent to the partial derivative of ξ with respect to *x*. Therefore by applying the principle of conservation of mass:

$$A_2 \rho_2 = A_0 \rho_0 (1+\varepsilon)^{-1} = A_0 \rho_0 \left(1 + \frac{\partial \xi}{\partial x}\right)^{-1} \quad and \quad A_1 \rho_1 = A_0 \rho_0 (1+\varepsilon_1)^{-1} \qquad (5)$$

Boundary condition I can now be defined by substituting equations (3) and (5) in (2)

.

$$(1+\varepsilon_1)^{-1} V_a = \left(1 + \frac{\partial \xi(0,t)}{\partial x}\right)^{-1} \frac{\partial \xi(0,t)}{\partial t} \qquad \text{Boundary condition I} \qquad (6)$$

Equation (6) may seem unusual for a boundary condition. But, it fits quite neatly into the subsequent analysis and produces results that correlate well with the results of the O.D.E. model. It is responsible for the mass transport that transfers tension from the previous span. This is an established feature of the O.D.E. model.

THE O.D.E. MODEL

Since the O.D.E. model has been confirmed by many years of use, it will be used to check the results of this analysis. Two versions are in use. One is nonlinear. It is used for cases such as startup of a process line where the web speed varies over a wide range. A linearized version is used for situations where the web speed changes by small amounts from a steady value. The linearized model will be used for the comparison. The linearized O.D.E. is:

$$L\frac{d\varepsilon_2(t)}{dt} = V_{a0}\varepsilon_1 - V_{b0}\varepsilon_2(t) + \left(V_b(t) - V_a(t)\right) \qquad \text{O.D.E. model} \tag{7}$$

where

$$\varepsilon_1 = \frac{T_1}{A_0 E} \quad and \quad \varepsilon_2(t) = \frac{T_2(t)}{A_0 E} \tag{8}$$

V_{a0} and V_{b0} are constant, nominal values of roller speed. $V_a(t)$ and $V_b(t)$ are small perturbations from V_{a0} and V_{b0}. For the purpose of this study $V_a(t)$ and ε_1 are held constant, $V_{a0} = V_{b0} = V_i$. $V_b(t)$ will be assumed to be a step input of magnitude, δv. The solution with these conditions is:

$$\varepsilon_2(t) = \varepsilon_1 + \frac{\delta v}{V_i}\left(1 - e^{-\frac{V_i}{L}t}\right) \qquad \text{O.D.E. solution} \tag{9}$$

At large time scales, where the propagation behavior of tension disturbances is invisible, the P.D.E. model should behave like equation (9).

THE P.D.E. MODEL

$$\frac{\partial^2 \xi}{\partial t^2} = C^2 \frac{\partial^2 \xi}{\partial x^2} \qquad \text{P.D.E.} \tag{10}$$

The one-dimensional wave equation (10) will be used to model the web. It is commonly seen in connection with the transverse oscillations of a fixed string or the longitudinal oscillation of a solid bar. In the case of longitudinal motion, the constant C is a function of the Young's modulus and density.

$$C = \sqrt{\frac{E}{\rho}} \tag{11}$$

Provided that the strain never becomes compressive, a string under tension may be treated as a solid bar. Equation (10) is based on two forces acting on particles of the string - inertial forces due to acceleration and elastic forces due to the spatial derivative of the strain. Its derivation can be found in most acoustics textbooks and will not be repeated here.

In the form shown above, the wave equation will not produce a solution corresponding to the situation described in Fig. 1. Application of the boundary conditions of the previous section will cause the portion of the web represented by the solution to move downstream. It will describe something like a flying carpet rather than a web moving over fixed rollers. The mathematical technique for handling this problem goes by many names - Euler description, Eulerian method, or Euler flux method. It is used in many contexts and is often misunderstood. Since it is central to this problem, its application will be described very explicitly.

THE EULER DESCRIPTION

Equation (10) is in a form known as the Lagrange description. It applies to a situation in which the parameter being calculated is associated with a point that is allowed to move, under the influence of physical laws, relative to the observer. This is the way normal way of thinking about dynamic problems. When applied to velocity or acceleration, it refers to what would be measured in the laboratory with a yardstick and stopwatch.

The alternative to the Lagrange description, (L.D.), is the Euler description, (E.D.). In the E.D. the equations describing the physics are modified so that the point of observation is held fixed as the material moves past. This is done by using the chain rule to explicitly separate the time and position derivatives. For example, if T represents temperature in a material that is moving along the *x* axis with transport velocity, V, the relationship between the Lagrange and Euler derivatives is:

$$\frac{DT}{Dt}=\frac{\partial T}{\partial t}+\frac{\partial T}{\partial x}V \qquad (12)$$

The Lagrange derivative is on the left. In fluid dynamics the term material derivative is used to emphasize that the Lagrange derivative is associated with a particular particle or piece of material. The first term on the right is the Euler derivative. It does not apply to a particular portion of material and includes none of the variation due to the transport motion. The last term adds the variation caused by transport motion.

Most discussions of the Euler description leave a number of questions unanswered.

1. Are E.D. variables "real"? For example, if web speed is measured at a fixed location using a friction wheel on a tachometer, is this an E.D. variable?
2. If the E.D. is used to derive a P.D.E., should the final step be to convert the solution back to a Lagrange description?
3. Should the auxiliary equations be be converted to an E.D. description?

The following example will provide insight into these questions. It does not involve a P.D.E. and, therefore, does not reveal all the effects of using (12). But, it shows clearly an important aspect of the Euler description - that it implies a change in variables. The key to understanding this is to avoid thinking about a moving coordinate system.

Imagine a long metal bar that is initially at rest and oscillating longitudinally with a standing wave of amplitude A. The variable ξ represents displacement of particles from their rest positions.

$$\xi(x,t)=A\cos(kx)\sin(\omega t) \qquad (13)$$

Now, assume that the bar is put into motion along its axis with transport velocity, v. Assuming that the variable, x refers to the same position in space as it did before, [in all that follows, no reference will be made to a moving coordinate system]. the new position of a particle in the bar will be:

$$l(x,t)=x+\mathrm{v}t+\xi(x,t)=x+\mathrm{v}t+A\cos(kx)\sin(\omega t) \tag{14}$$

The particle velocity for the moving bar is found by taking the time derivative of (14) :

$$v(x,t)=\mathrm{v}+A\omega\cos(kx)\cos(\omega t) \tag{15}$$

Equations (14) and (15) illustrate the flying carpet problem. Substituting x = x_o and t = t_o into (15) does not produce the velocity of a particle located at x_o. The location of the particle with this velocity is:

$$l(x_o,t_o)=x_o+\mathrm{v}t_o+A\cos(kx_o)\sin(\omega t_o) \tag{16}$$

Equation (16) applies to a point that has moved along with the bar by an amount, $\mathrm{v}t_o$.

Replacing the x variable with $x-\mathrm{v}t$ solves the problem. When this is done, equations (15) and (14) become:

$$v'(x,t)=\mathrm{v}+A\omega\cos(k(x-\mathrm{v}t))\cos(\omega t) \tag{17}$$

$$l'(x,t)=x+A\cos(k(x-\mathrm{v}t))\sin(\omega t) \tag{18}$$

Now, the original $\mathrm{v}t_o$ term in (16) has been eliminated and $-\mathrm{v}t_o$ in the cosine term shifts the standing wave forward by that amount. Thus, the point of observation effectively remains fixed at x o while the bar moves forward.

Equations (17) and (18) produce the correct values for velocity and position. But now, there is a problem with the time derivatives. If one attempts to calculate velocity by differentiating (18) with respect to time, the result is:

$$\frac{\partial\, l'(x,t)}{\partial t}=Ak\mathrm{v}\sin(k(x-\mathrm{v}t))\sin(\omega t)+A\omega\cos(k(x-\mathrm{v}t))\cos(\omega t) \tag{19}$$

This is clearly not the same as (17). The first term shouldn't be there and the v term is missing. This is what the Euler description corrects. Because:

$$\frac{\partial\, l'(x,t)}{\partial x}\mathrm{v}=\mathrm{v}\;1-A\sin(k(x-\mathrm{v}t))\sin(\omega t) \tag{20}$$

And, when this term is added to (19) the velocity of (15) reappears.

$$\frac{\partial\, l'(x,t)}{\partial t}+\frac{\partial\, l'(x,t)}{\partial x}\mathrm{v}=\mathrm{v}+A\omega\cos(k(x-\mathrm{v}t))\cos(\omega t) \tag{21}$$

Thus, the Euler description can be interpreted as an adjustment resulting from change in the x variable to x - vt.

Mathematicians may feel better if the example is generalized. Let,

$$\xi_L = f\,(x,t) \qquad \text{Lagrange description} \qquad (22)$$

$$\xi_E = f\ \left(x - \mathrm{v}t\right), t \qquad \text{Euler description} \qquad (23)$$

If (22) and (23) are viewed as a straightforward change of variable, then, by the chain rule:

$$\frac{\partial \xi_E}{\partial t} = \frac{\partial \xi_L}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial \xi_L}{\partial x}\frac{\partial\left(x - \mathrm{v}t\right)}{\partial t} = \frac{\partial \xi_L}{\partial t}1 + \frac{\partial \xi_L}{\partial x}(0 - \mathrm{v}) = \frac{\partial \xi_L}{\partial t} - \frac{\partial \xi_L}{\partial x}\mathrm{v} \qquad (24)$$

Using the chain rule again:

$$\frac{\partial \xi_E}{\partial x} = \frac{\partial \xi_L}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial \xi_L}{\partial x}\frac{\partial\left(x - \mathrm{v}t\right)}{\partial x} = \frac{\partial \xi_L}{\partial t}0 + \frac{\partial \xi_L}{\partial x}(1 - 0) = \frac{\partial \xi_L}{\partial x} \qquad (25)$$

So, using (25) in (24) leads to:

$$\frac{\partial \xi_L}{\partial t} = \frac{\partial \xi_E}{\partial t} + \frac{\partial \xi_E}{\partial x}\mathrm{v} \qquad (26)$$

The answers to the questions posed at the beginning of this discussion are now obvious.

1. Real world measurements are always Lagrangian. The Euler description is just a mathematical artifice associated with the use of partial derivatives under certain circumstances. This is true even if it produces terms that have meaning in other contexts. An example of this is the mixed derivative in equation (30). In the study of transverse oscillations it can be interpreted as the Coriolis acceleration. However, this does not change the main purpose of the technique - which is to facilitate the solution of a problem in which the point of observation is held fixed as the material moves past.
2. If a P.D.E. is converted to an Euler description and then solved for a time derivative, such as velocity, the solution should be converted back to a Lagrange description using (12).
3. The auxilliary equations must be subjected to the same change in variables as the P.D.E.

A professional mathematician may find this exercise to be a tiresome illustration of the chain rule. I include it here because 1) I have not seen the Euler description described this way in the literature and 2) I found it essential to understanding the physical meaning of the operations of the next section.

It should be noted that the effect of the Euler description on a P.D.E. goes beyond replacing the x coordinate with $x-\mathrm{v}t$. It changes the very nature of the problem. In this instance it creates a model in which the web is moving, yet has fixed boundaries.

THE EULER DESCRIPTION P.D.E.

Two changes will be made in the problem variables. First, they will be separated into a large steady value plus a small varying component. Second, an Euler description will be adopted.

The longitudinal velocity of web particles will be assumed to consist of two parts - the axial transport velocity, V_i plus a varying component, $\frac{\partial \xi'}{\partial t}$.

Corresponding to the two velocities, there will be two components of strain - a constant component, ε_1 plus a varying component $\frac{\partial \xi'}{\partial x}$.

The Euler description is applied to the varying component of the velocity.

$$\frac{\partial \xi'}{\partial t}=\frac{\partial \xi_E}{\partial t}+V_i\frac{\partial \xi_E}{\partial x}.$$

So the complete transformation of variables is:

$$\frac{\partial \xi_L}{\partial t}=V_i+\frac{\partial \xi'}{\partial t}=V_i+\frac{\partial \xi_E}{\partial t}+V_i\frac{\partial \xi_E}{\partial x} \tag{27}$$

$$\frac{\partial \xi_L}{\partial x}=\varepsilon_1+\frac{\partial \xi_E}{\partial x} \tag{28}$$

The "E" and "L" subscripts identify variables as Euler or Lagrange. The "E" subscript will be understood to encompass the separation of constant and varying components as well as the Euler description. This practice will be followed throughout the remainder of this paper.

Substituting (27) and (28) into (10):

$$\frac{\partial\left(V_i+\frac{\partial \xi_E}{\partial t}+V_i\frac{\partial \xi_E}{\partial x}\right)}{\partial t}+V_i\frac{\partial\left(V_i+\frac{\partial \xi_E}{\partial t}+V_i\frac{\partial \xi_E}{\partial x}\right)}{\partial x}=C^2\frac{\partial\left(\varepsilon_1+\frac{\partial(\xi_E)}{\partial x}\right)}{\partial x} \tag{29}$$

Performing the operations indicated in (29) produces the P.D.E. model.

$$\frac{\partial^2 \xi_E}{\partial t^2}+2V_i\frac{\partial^2 \xi_E}{\partial x\partial t}+\frac{\partial^2 \xi_E}{\partial x^2}V_i^2=C^2\frac{\partial^2 \xi_E}{\partial x^2} \qquad \text{The E.D. P.D.E.} \tag{30}$$

Equation (30) was presented in one of the earliest traveling string papers by Sack [2]. It is the most common form of the one-dimensional wave equation for a moving medium.

It is important to keep in mind that any solution of (30) for a time derivative must be transformed back to a Lagrange description before comparing it to laboratory results.

CONVERSION OF THE BOUNDARY AND INITIAL CONDITIONS TO AN EULER DESCRIPTION

The equations will be presented first in their most natural form - the Lagrange description. Then, equations (27) and (28) will be used to transform them into an Euler description.

At time zero the web will be assumed to be running at a uniform speed of V_i and strain ε_1. This implies that the circumferential velocity of roller A is V_i. The circumferential velocity or roller B is assumed to be V_i plus a forcing function $f(t)$ beginning at time zero.

First, the Lagrange versions:

$$(1+\varepsilon_1)^{-1}\left(1+\frac{\partial\xi_L(0,t)}{\partial x}\right)V_i=\frac{\partial\xi_L(0,t)}{\partial t} \qquad \text{Boundary condition I, (L.D.)} \quad (31)$$

$$\frac{\partial\xi_L(L,t)}{\partial t}=V_i+f(t) \qquad \text{Boundary condition II, (L.D.)} \quad (32)$$

$$\frac{\partial\xi_L(x,0)}{\partial t}=V_i \qquad \text{Intital condition I, (L.D.)} \quad (33)$$

$$\frac{\partial\xi_L(x,0)}{\partial x}=\varepsilon_1 \qquad \text{Initial condition II, (L.D.)} \quad (34)$$

$$\xi_L(x,0)=\varepsilon_1 x \qquad \text{Initial condition III, (L.D.)} \quad (35)$$

$$\xi_L(0,0)=0 \qquad \text{Initial condition IV, (L.D.)} \quad (36)$$

$$\xi_L(L,0)=\varepsilon_1 L \qquad \text{Initial condition V, (L.D.)} \quad (37)$$

The initial conditions II, III and V need a little justification. The particular problem that is going to be analyzed assumes initial conditions corresponding to a web with initial strain ε1. The initial displacements must exist as a consequence of that strain. If this seems odd, think about the case of transverse displacements in a fixed string.

Now for the Euler versions:

$$V_i\frac{-\varepsilon_1}{(1+\varepsilon_1)}\left(\frac{\partial\xi_E(0,t)}{\partial x}\right)=\frac{\partial\xi_E(0,t)}{\partial t} \qquad \text{Boundary condition I, (E.D.)} \quad (38)$$

$$\frac{\partial \xi_E(L,t)}{\partial t}+V_i\frac{\partial \xi_E(L,t)}{\partial x}=f(t)$$ Boundary condition II, (E.D.) (39)

$$\frac{\partial \xi_E(x,0)}{\partial t}=0$$ Initial condition I, (E.D.) (40)

$$\frac{\partial \xi_E(x,0)}{\partial x}=0$$ Initial condition II, (E.D.) (41)

$$\xi_E(x,0)=\frac{\partial \xi_E(x,0)}{\partial x}x=0$$ Initial condition III, (E.D.) (42)

$$\xi_E(0,0)=0$$ Initial condition IV, (E.D.) (43)

$$\xi_L(L,0)=0$$ Initial condition V, (E.D.) (44)

THE SOLUTION

Laplace transforms will be used to integrate the P.D.E. The solution follows a procedure described by Churchill in his book "Operational Mathematics" [14].

The first step is to take the time transform of the P.D.E.,(30).

$$s^2\boldsymbol{L}\xi_E(x,t)-s\frac{\partial \xi_E(x,0)}{\partial t}-s\xi_E(x,0)+2V_i\left[\frac{\partial\left(s\boldsymbol{L}\xi\text{-}\xi(x,0)\right)}{\partial x}\right]=\left(C^2-V_i^{\,2}\right)\boldsymbol{L}\frac{\partial^2\xi_E(x,t)}{\partial x^2} \quad (45)$$

The next steps will be clearer if the following change of variable is made.

$$U(x,s)=\boldsymbol{L}\xi_E(x,t) \quad (46)$$

The last term of (45) can be modified to facilitate the analysis. It can be shown that an interchange in the order of differentiation with respect to x and integration with respect to t leaves the value unchanged. Thus,

$$\boldsymbol{L}\frac{\partial^2\xi E(x,t)}{\partial x^2}=\frac{\partial^2\boldsymbol{L}\xi_E(x,t)}{\partial x^2}=\frac{\partial^2 U}{\partial x^2} \quad (47)$$

Substituting (46), (47), (41) and (42) in (45) produces a relationship that can be treated as an ordinary differential equation in one variable.

$$s^2U+2V_is\frac{\partial^2U}{\partial x^2}=\left(C^2-V_i^{\,2}\right)\frac{\partial^2U}{\partial x^2} \quad (48)$$

The solution of (48) is:

$$U=c_1e^{\alpha_1 sx}+c_2e^{\alpha_2 sx} \quad (49)$$

$$\alpha_1 = \frac{1}{C - V_i} \qquad \alpha_2 = \frac{1}{C + V_i} \tag{50}$$

For this problem, strain is more important than displacement. To obtain strain, (49) is differentiated with respect to x.

$$\frac{\partial \mathbf{L}\xi_E(x,t)}{\partial x} = c_1 s\alpha_1 e^{\alpha_1 sx} + c_2 s\alpha_2 e^{\alpha_2 sx} \tag{51}$$

The inverse transform of (51) solves the problem provided c_1 and c_2 can be evaluated. They are found from the boundary conditions.

Taking the time transform of boundary condition I:

$$V_i \frac{-\varepsilon_1}{1+\varepsilon_1} \frac{\partial \mathbf{L}\xi_E(x,t)}{\partial x}\bigg|_{x=0} = s\mathbf{L}\xi_E(x,t) + \xi_E(x,0)\Big|_{x=0} \tag{52}$$

Substituting (49), (51) and (43) in (52) with $x = 0$ produces:

$$-V_i \frac{\varepsilon_1}{1+\varepsilon_1}(c_1\alpha_1 s + c_2\alpha_2 s) = s(c_1 + c_2) \tag{53}$$

Taking the time transform of boundary condition II:

$$V_i \frac{\partial \mathbf{L}\xi_E(x,t)}{\partial x}\bigg|_{x=L} + s\mathbf{L}\xi_E(x,t) + \xi_E(x,0)\Big|_{x=L} = f(t) \tag{54}$$

Substituting (49), (51) and (44) in (54) with x = L produces:

$$V_i\left(c_1\alpha_1 s e^{\alpha_1 sL} + c_2\alpha_2 s e^{\alpha_2 sL}\right) + s\left(c_1 e^{\alpha_1 sL} + c_2 e^{\alpha_2 sL}\right) = \mathbf{L}f(t) \tag{55}$$

Equations (53) and (55) can now be solved simultaneously for c1 and c2. Substituting these values along with (50) into (51) produces:

$$\frac{\partial \xi_E}{\partial x} = \mathbf{L}^{-1}\left\{ \frac{\dfrac{\mathbf{L}f(t)}{c}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i} - \frac{sL}{C-V_i}} \right]}{1 - \beta e^{-\frac{2sLC}{C^2 - V_i^2}}} \right\} \tag{56}$$

where

$$\beta = \frac{-V_i + (\varepsilon_1 + 1)C}{V_i + (\varepsilon_1 + 1)C} \tag{57}$$

The denominator of (56) prevents a straightforward use of a table of transforms. In a similar situation Churchill [14] uses the following series expansion.

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n \qquad 0 < z < 1 \tag{58}$$

β and $e^{-\frac{2sLC}{C^2-V_i^2}}$ are both less than 1. (There is no requirement to consider the transform variable, s as complex in this problem. Therefore, it can be considered as real and positive). Applying (58), to equation (56) converts it to a form that is easily inverted.

$$\frac{\partial \xi_E}{\partial x} = \boldsymbol{L}^{-1}\left\{\frac{\boldsymbol{L}f(t)}{c}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i}-\frac{sL}{C-V_i}}\right]\sum_{n=0}^{m} e^{-\frac{2nLCs}{C^2-V_i^2}}\beta^n\right\} \tag{59}$$

The final step is to return to a Lagrange description by applying equation (28).

$$\frac{\partial \xi_L}{\partial x} = \varepsilon_1 + \boldsymbol{L}^{-1}\left\{\frac{\boldsymbol{L}f(t)}{c}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i}-\frac{sL}{C-V_i}}\right]\sum_{n=0}^{m} \beta^n e^{-\frac{2nLCs}{C^2-V_i^2}}\right\} \tag{60}$$

Equation (60) will now be solved for a variety of driving functions.

STEP FUNCTION INPUT

The first driving function to be analyzed will be the unit step, $\Phi(t)$. The step starts at t = 0 and adds a small increment, δv to the initial velocity V_i.

$$f(t) = \delta v\ \Phi(t) \tag{61}$$

The transform of f(t) is:

$$\boldsymbol{L}f(t) = \frac{\delta v}{s} \tag{62}$$

Substituting (62) into(60) and inverting produces the following solution.

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\left[\begin{array}{l}\beta^n \Phi\left(\dfrac{x-V_i t+Ct}{C-V_i} - \dfrac{L}{C-V_i} - \dfrac{2nLC}{C^2-V_i^2}\right) + \\ \beta^{n+1}\Phi\left(\dfrac{-(x-V_i t)+Ct}{C+V_i} - \dfrac{L}{C-V_i} - \dfrac{2nLC}{C^2-V_i^2}\right)\end{array}\right] \tag{63}$$

For a given value of t, this is a finite series because the time-shifted unit step functions (Heaviside functions) are replaced by zero when their arguments become negative. So, for t less than some t_{max}:

$$m = t_{\max} \frac{C^2 - V_i^2}{2LC} \qquad \text{rounded up to the nearest integer} \qquad (64)$$

INTERPRETATION OF EQUATION (63)

Although equation (63) is straightforward for purposes of calculation, it is hard to visualize. The diagram in Fig. 2 may help.

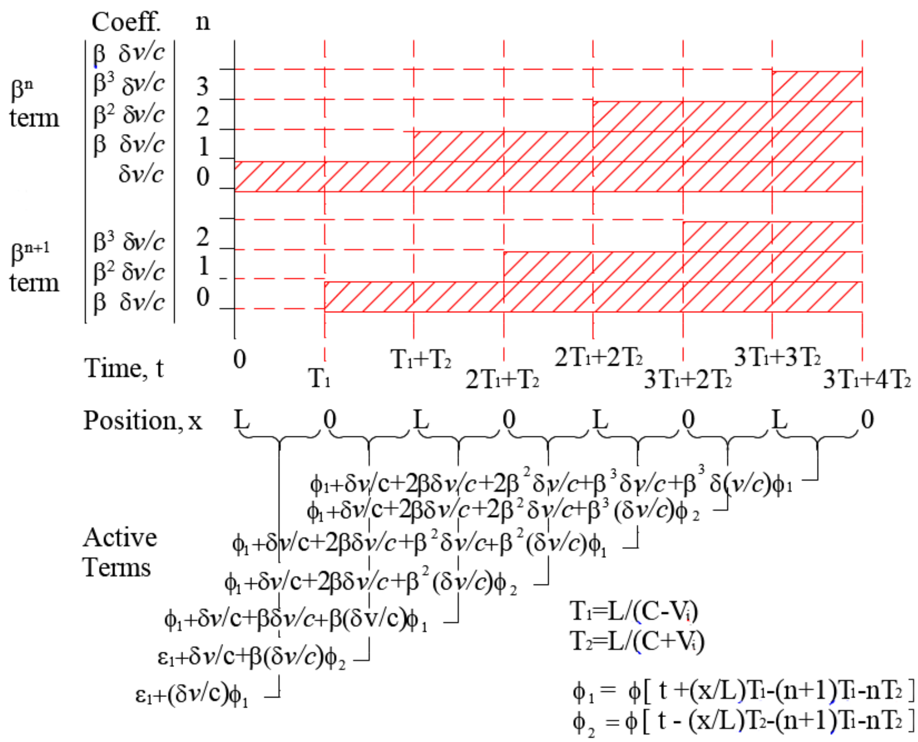

Figure 2 - A diagram of equation (63)

The bars in the chart can be viewed as time-shifted step functions. The bars are grouped vertically to correspond to the two terms of equation of (63). The bottom bar in each group represents the step function for the summation index, n = 0. The next bar up is for n = 1, etc. The leftmost column shows the amplitude coefficient of each step. On the horizontal axis there are two scales. One is for time. The other is for position. The position axis starts at $x = L$ goes to $x = 0$ and then back to L in a repeating pattern. The time scale advances along with x at a rate consistent with the propagation velocity, $C+V_i$ or $C\text{-}V_i$. In the bottom third of the chart, equations show how the summation progresses with time. Each series is formed by adding up the terms indicated by the bars in its respective column. For example, during the time interval from $2T_1+T_2$ to $2T_1+2T_2$ the leading edge of the disturbance is in the process of moving from 0 to L. It has already gone through three previous cycles, advancing from L to 0, 0 to L, L to 0 again. In each cycle the strain grows by an amount shown by the last term in the series.

It will be noticed that the amplitude of the disturbance changes slightly on reflection at $x=0$. It changes by the ratio of β^{n+1}/β^n. At x = L the ratio is 1.

COMPARISON OF RESULTS WITH THE O.D.E. MODEL

Comparison of the P.D.E. model (63) with the O.D.E. model of equation (9) shows that on large time scales (large compared to the time for disturbances to propagate through the span) they behave alike. Graphs in Figs. 3 and 4 illustrate an example. Parameters for both models are shown below (in the P.D.E. model $x=L$).

V_i = 10 m/sec c = 1500 m/sec L = 1 m ε_1 = .0005 δv = .0001*V_i

Fig. 3 shows the step response of the P.D.E. model. Fig. 4 shows the percent difference between the two. On this time scale they are in very close agreement. The graph in Fig. 4 appears solid because the error makes a cycle once every 1.33 milliseconds. This is due the stair-like behavior of the P.D.E solution.

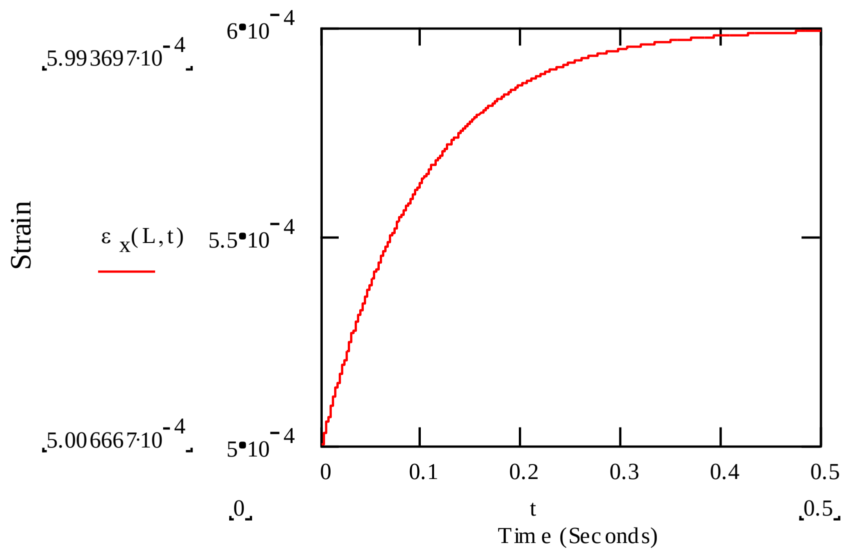

Figure 3 - Step response of the P.D.E. at $x=L$, V_i = 10 m/sec, C = 1500 m/sec, L = 1 m, ε_1 = .0005, δv = .0001*V_i

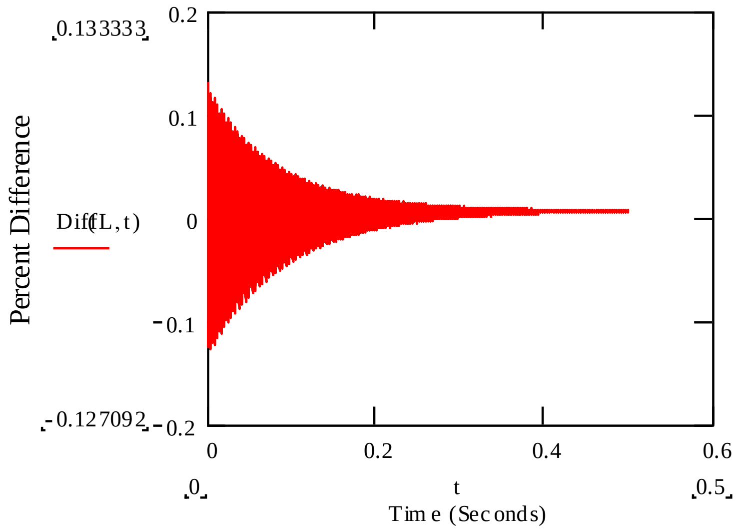

Figure 4 - Difference between the O.D.E. and the P.D.E. solutions

Fig. 5 shows a portion of the P.D.E. model data (solid line) at higher resolution. It is superimposed on the O.D.E. data (dashed line). The stair-like shape of the P.D.E. graph is due to the propagation delay. The strain disturbance initiated at roller B travels through the web span toward roller A. The strain doesn't change at a particular point in the span until the disturbance reaches it at intervals of 1.33 milliseconds. In this case the P.D.E. data is shown at $x = L$.

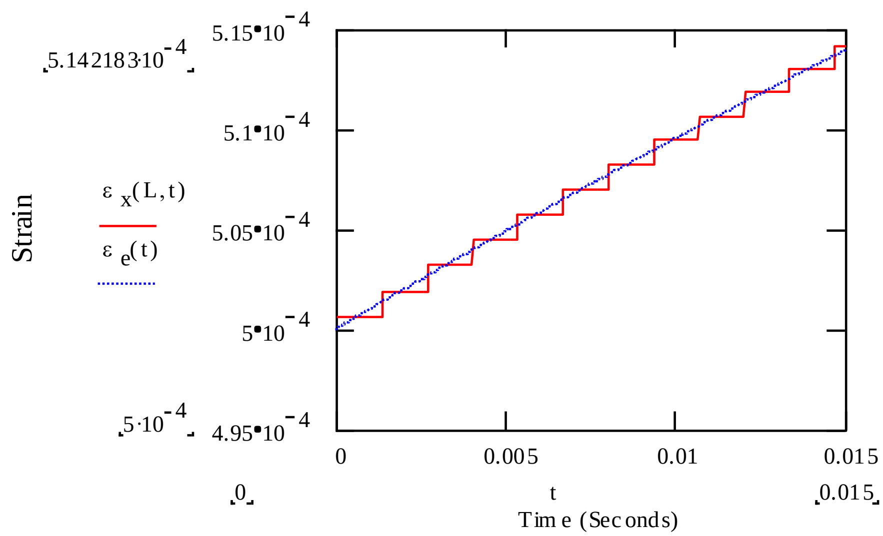

Figure 5 - High-resolution view of the first .015 seconds of the P.D.E (solid line) and O.D.E. (dashed line) solutions.

Fig. 6 is shows a different view of the solution. The abscissa is distance along the span instead of time. The disturbance is shown at four different times. It starts at $x = 1$ meter and progresses to the left until it reaches the end at $x = 0$ where it is reflected. It takes .671 milliseconds to travel this distance. The upper ramp at the left end has been reflected and is moving back to the source. It will take .662 milliseconds to make the return trip. This action continues with the strain rising in progressively smaller increments on each cycle until it reaches its steady state value of $\delta V_i/V_i + \varepsilon_1$. The velocity of the disturbance is equal to C-V_i traveling upstream and C+V_i downstream.

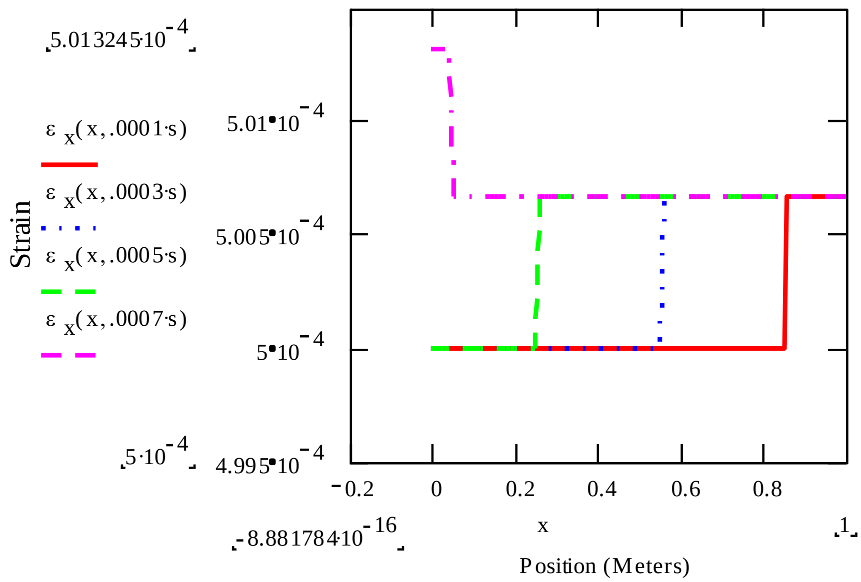

Figure 6 - Progress of a strain step-disturbance along the span. Strain is shown at .1, .3, .5 and .7 milliseconds. It starts at x = 1 meter and progresses to the left until it reaches the end at x = 0 where it is reflected. It takes .671 milliseconds to travel this distance. The upper ramp at the left end has been reflected and is moving back to the source. It will take .662 milliseconds to make the return trip.

EXPONENTIAL BEHAVIOR OF P.D.E. (63)

While the example in the previous section is very strong evidence for agreement between the P.D.E. and the O.D.E., it is not proof. What is needed is to demonstrate mathematically that if C is allowed to become arbitrarily large, equation (63) becomes equivalent to (9).

The first step is to show that for large values of C.

$$\beta^n = \left(\frac{-V_i + (\varepsilon_1 + 1)C}{V_i + (\varepsilon_1 + 1)C}\right)^n \cong e^{-(something)} \qquad (65)$$

Taking the natural log of the left side of the previous expression:

$$\ln\left(\beta^n\right)=n\ln\left(\frac{\frac{V_i}{C}+1+\varepsilon_1}{\frac{V_i}{C}+1+\varepsilon_1}\right)^n=n\left[\ln\left(\frac{-V_i}{C}+1+\varepsilon_1\right)-\ln\left(\frac{V_i}{C}+1+\varepsilon_1\right)\right] \tag{66}$$

The quantity ln(1 + z) may be replaced by z, because:

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots\cong z \qquad if\ \ z<<1 \tag{67}$$

So, expression (66) becomes:

$$\ln\left(\beta^n\right)\cong n\left(\frac{-2V_i}{C}\right) \tag{68}$$

$$\therefore \qquad \beta^n \cong e^{-2n\frac{V_i}{C}} \tag{69}$$

Equation (69) can be converted to time by replacing n with:

$$n\cong\frac{t}{2L/c} \tag{70}$$

Equations (69) and (70) can now be used with (63)to calculate the shape of the amplitude envelope.

$$\frac{\delta V_i}{c}\sum_{k=0}^{n}\beta^k(1+\beta)\cong\frac{\delta V_i}{c}\sum_{k=0}^{n}2\beta^k\cong 2\,\frac{\delta V_i}{c}\int_0^n e^{-2n\frac{V_i}{c}}dn=\frac{\delta V_i}{V_i}\left(1-e^{-\frac{V_i}{L}t}\right) \tag{71}$$

Substituting (70) into (69) produces another relationship that will be useful later.

$$\beta^n=\beta^{-\frac{t}{2L/c}}\cong e^{-\frac{V_i}{L}t} \tag{72}$$

RESPONSE TO A SINGLE PULSE

What happens when a single velocity pulse occurs at roller B? In the case of a fixed string without any damping, the pulse would travel back and forth between the supports indefinitely. Intuition suggests that the traveling string will be different because material is flowing out of the span at roller B and being replaced with new material at roller A. To answer this question a pulse of amplitude δv and length t_1 will be used in equation (60). The transform for this is:

$$\boldsymbol{L}f(t)=\delta v\frac{1-e^{-t_1 s}}{s} \tag{73}$$

Substituting in (60) and inverting:

$$\frac{\partial \xi}{\partial x}(x,t)=\varepsilon_1+\frac{\delta v}{C}\sum_n\left[\begin{array}{l}\beta^n\left[\begin{array}{l}\Phi\left(\frac{x-V_i t+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)-\\ \Phi\left(\frac{x-V_i t+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}-t_1\right)\end{array}\right]\\ +\beta^{n+1}\left[\begin{array}{l}\Phi\left(\frac{-(x-V_i t)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)-\\ \Phi\left(\frac{-(x-V_i t)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}-t_1\right)\end{array}\right]\end{array}\right] \tag{74}$$

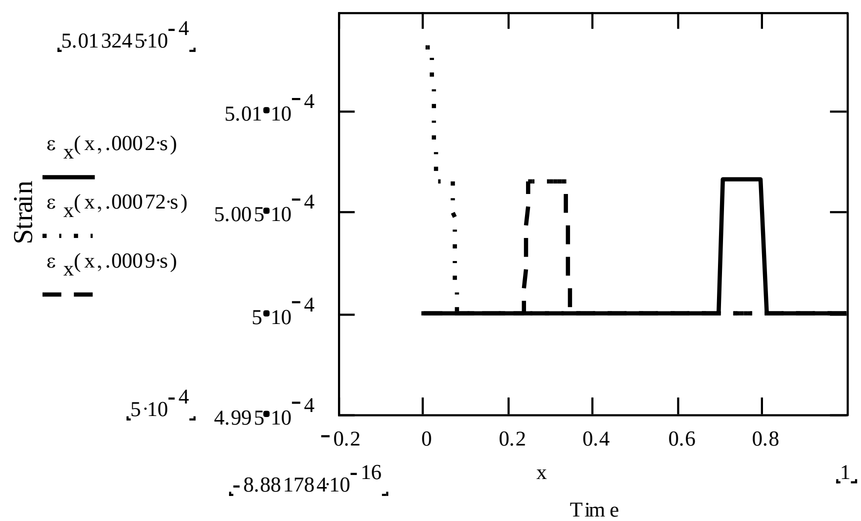

Fig. 7 - A single pulse at three different times. It advances in the same manner as the step input. The first pulse on the right has just left roller B. The pulse on the left shows the same pulse being reflected from roller A. The pulse in the middle shows it after it is reflected and is returning to B.

The results are illustrated in Figs. 7 and 8, using parameters similar to the those for the step input. Fig. 7 shows a single pulse at three different times. It advances in the same manner as the step input, going from roller B to roller A, where it is reflected back to B. Then, it is reflected again and the cycle is repeated. Fig. 8 shows the pulse amplitude envelope over one second. It changes as expected. An investigation, using results from the previous section, shows that it decays exponentially from an initial amplitude of $\delta v/C$. At the ends of the span it momentarily doubles during reflection. The time constant is L/V_i.

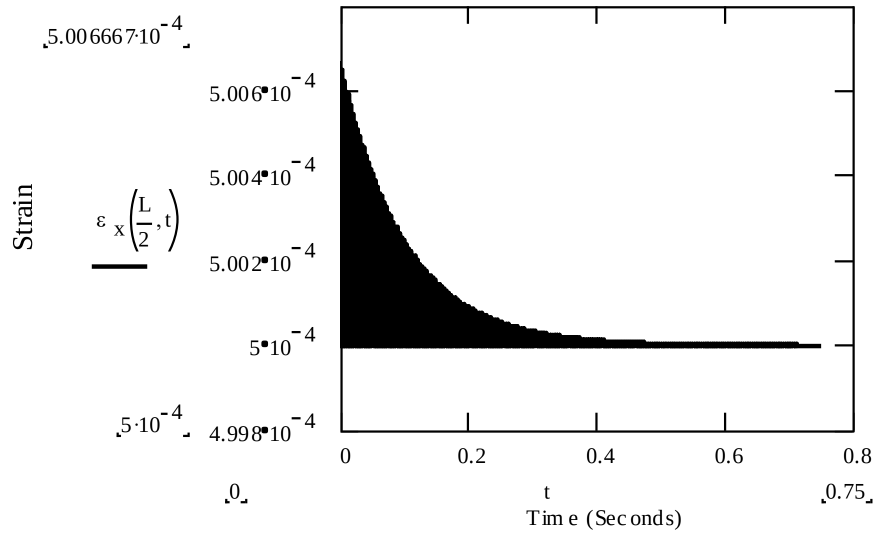

Fig. 8 - Decay envelope of a single pulse over aperiod of 1 second

A descriptive term for this decay is transport damping.

Decay of the pulse illustrates an important feature of wave propagation in materials moving over fixed supports. Energy in the material between the supports is not conserved. Miranker [11], using an ingenious technique, was the first to point out that this happened for transverse oscillations. Wickert and Mote [15] later analyzed the phenomenon and showed the energy transfer involved the supports. It is evident that similar conclusions apply to longitudinal strain variations. This is easy to see if one thinks about what happens while a pulse is being reflected at a roller. At that time the roller experiences a change in torque due to the tension change. And, since the roller is rotating, work is done, either on the roller or on the web. Additionally, some energy is transferred downstream due to tension transfer across roller B.

In addition to the decay predicted by this idealized model, real web materials will have viscoelastic damping. So, it is safe to assume that a single, short pulse will be attenuated quickly.

RESPONSE TO REPETITIVE PULSES

The solution for this problem follows a slightly different pattern than with the other inputs. The Laplace transform for a repetitive pulse train of amplitude δv, period t_2 and pulse length, t_1 is:

$$\boldsymbol{L}f(t) = \frac{\delta v}{s}\frac{1-e^{-st_1}}{1-e^{-st_2}} \tag{75}$$

The denominator of the forcing function requires the same treatment as in (56). Using (58) a second time leads to a double summation.

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\beta^n\sum_{k=0}^{p}\left[\left[\Phi\left(\frac{x-V_it+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)-kt_2 \atop \Phi\left(\frac{x-V_it+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}-kt_2-t_1\right)\right] + \beta\left[\Phi\left(\frac{-(x-V_it)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)-kt_2 \atop \Phi\left(\frac{-(x-V_it)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}-kt_2-t_1\right)\right]\right] \quad (76)$$

$$p=\frac{t_{\max}}{t_2} \qquad \text{rounded up} \quad (77)$$

A particularly interesting case arises when the repetition period is equal to the propagation delay, $L/(C-V_i)+L/(C+V_i)$. Then, each new pulse is met by the reflection of the one before. The amplitude reduction due to transport damping is greatly exceeded by the reinforcement of the new pulse. Using such a pulse train, with the other parameters the same as the previous example, leads to the result illustrated in Fig, 9. The pulses grow exponentially to an amplitude of $\delta V_i/V_i$ with a time constant of L/V_i . The final amplitude is the same as if the pulse had been a step.

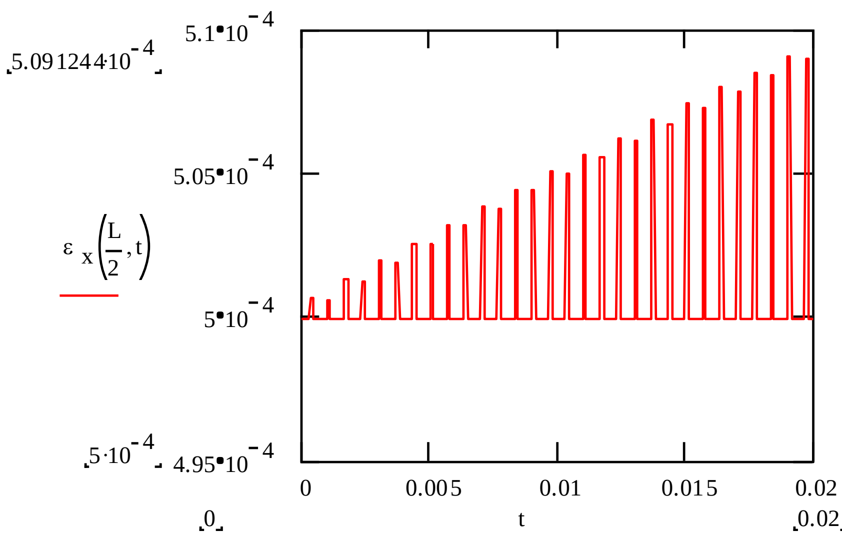

Figure 9 - Repetitive pulse with period = *L/(C-Vi)+L/(C+Vi).* The pulses grow exponentially to an amplitude of $\delta V_i/V_i$ with a time constant of L/V_i . The final amplitude is the same as if the pulse had been a step. For this example: V_i = 10 m/sec , C = 1500 m/sec, L = 1 m, ε_1 = .0005, δv = .0001*V_i , t_1 = .000067 sec $t_2 = L/(C-V_i)+L/(C+V_i)$ = .00133 *sec.*

This clearly has implications for a web process. An eccentric or unbalanced roller could produce a disturbance once each revolution. If the web speed and span length are such that the period of the disturbance is an integer fraction (or if damping is low, an integer multiple) of $L/(C-V_i)+L/(C+V_i)$ the pulse may be amplified. Even a pulse which is attenuated by the viscoelastic damping of the web material may be amplified to many times that of a single pulse.

SINUSOIDAL INPUT

The transform for a sinusoidal input is:

$$\boldsymbol{L}f(t)=\frac{\delta V_i \omega}{s^2+\omega^2} \tag{78}$$

Substituting in (60) and inverting:

$$\frac{\partial \xi_L}{\partial x}(x,t)=\varepsilon_1+\frac{\delta v}{C}\sum_{n=0}^{m}\beta^n \begin{bmatrix} \sin\left[\omega\left(\frac{x-V_i t+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}\right)\right]\bullet \\ \Phi\left(\frac{x-V_i t+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}\right)+ \\ \beta\sin\left[\omega\left(\frac{-(x-V_i t)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}\right)\right]\bullet \\ \Phi\left(\frac{-(x-V_i t)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}\right) \end{bmatrix} \tag{79}$$

The input will reinforce itself in the same manner as a repetitive pulse when:

$$\omega=2\pi n\frac{\left(C^2-V_o^2\right)}{2LC} \qquad n = 1, 2, 3 \cdots \tag{80}$$

At these frequencies the amplitude behaves in a manner similar to repetitive pulses. It grows exponentially to an amplitude of $\delta V_i/V_i$ with a time constant of L/V_i.

HIGH SPEED BEHAVIOR

As the transport speed, V_i approaches C, equation (80) approaches 0 for all n. Also, the upstream propagation velocity, $C-V_i$ approaches 0. This clearly indicates that something unusual happens at $V_i = C$. Could one see a standing wave of zero frequency? Study of the traveling string literature suggests that a more sophisticated, nonlinear model is needed at these speeds. Furthermore, there are many other phenomena that will become significant as speeds increase. The answer to this question should be postponed pending further study.

CONCLUSIONS

This highly idealized model has two principal uses. First, it is a necessary step toward more realistic models. Second, it provides a framework for understanding tension problems whose causes may have been unrecognized in the past.

Some of the shortcomings in the present model are:

1. There is no provision for the variation in mass per unit length in the span. This is important for modeling mass flow and is provided only at the upstream boundary in the present model. This can be done. But, it leads to a nonlinear equation requiring numerical methods for its solution. Lack of this feature will probably not affect the general behavior at low speeds.
2. There is no provision for viscoelasticity. This will obviously have a strong effect on the shape, velocity (dispersion) and amplitude of disturbances in polymer materials. Metals will be much less affected. This can also be modeled but it requires a higher order nonlinear equation.

3. No testing has been done.

The model suggests that the following phenomena may be seen in real applications. They will undoubtedly be attenuated and distorted by viscoelasticity, nonlinearities and friction. But, they will probably be observable.

1. The strain pulse produced by a brief (of the order of L/(C-Vi)+L/(C+Vi) speed difference between two rollers will be very small. The amplitude will be of the order of Vi/C times the amplitude that would be produced if the speed difference were present continuously.

2. The velocity of a disturbance is equal to C-Vi traveling upstream and C+Vi downstream. C may be a function of wavelength due to nonlinearities and viscoelasticity. But, for a single wavelength the relationship holds.

3. When the difference in web speed between two rollers is increased rapidly to a new steady value, the initial strain will be only of the order of V_i/C. This change will travel at the speed of sound to the nearest roller where it will almost double in size and be reflected back toward the source. At the source it will be reflected again. This action will continue, with the strain rising in progressively smaller increments on each cycle, until it reaches a steady state value of $\delta v/V_i$. The rise will be approximately exponential with a time constant of L/V_i . At large time scales, where the steps are imperceptible, the behavior will match the O.D.E. model.

4. A single brief pulse (shorter than the time for the pulse to travel up the span and back again) will be damped by the transport motion. The pulse will be reflected back and forth. But, unlike a pulse in an ideal fixed string, it will decay a little each cycle until it disappears. This "transport damping" along with viscoelasticity and friction will help remove energy from disturbances. An example of such a pulse is a sudden slip on a roller due to passage of a wrinkle or a splice.

5. A repetitive disturbance can be amplified if the period is an integer fraction of $L/(C-V_i)+L/(C+V_i)$. If damping is low it may be amplified even at integer multiples of the

delay time. It will start with low amplitude and grow exponentially. The time constant will be L/V_i. The final amplitude will be the same as if the pulse had been a step. An example of a repetitive pulse source is the cyclic disturbance of an embossing roller.

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