A Unified Model for Longitudinal and Lateral Web Dynamics

Jerry Brown

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The basic idea

- At the 2017 conference, I showed that Interaction between the normal entry rule and mass transfer between spans can be used to derive Shelton’s Dynamic beam model.
- This paper generalizes the mass transfer idea and uses it to develop a dynamic model that combines lateral and longitudinal (tension) behavior.

Proof that it works

- Results from the unified model will be shown to agree closely (very closely) with the static and dynamic beam models developed and tested by Shelton for his 1968 dissertation.
- It also agrees closely with the Timoshenko model I presented at the 2017 conference.

New results

- Some new things will be shown.
  - The effect of roller motion on tension
  - A new source of wrinkling – gradients in lateral velocity
  - The effect of tension change on lateral position

The model

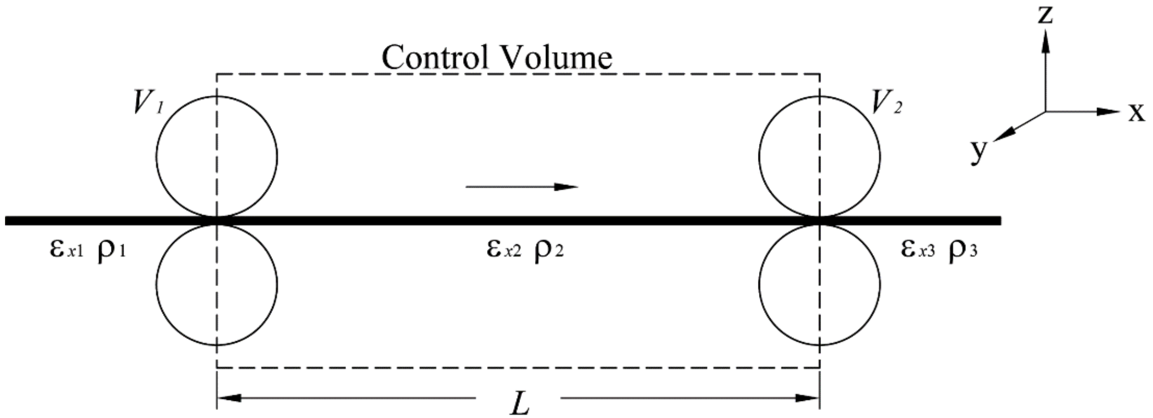
- Based on nonlinear elasticity theory
  - Nonlinear elasticity is necessary because of the effect of MD tension on the elastic curve of the web.

Nonlinear Equations of equilibrium

$$\frac{\partial}{\partial x}\left[\sigma_{xx} - \omega_z \tau_{xy}\right] + \frac{\partial}{\partial y}\left[\tau_{xy} - \omega_z \sigma_{yy}\right] = 0$$

$$\frac{\partial}{\partial y}\left[\sigma_{yy} + \omega_z \tau_{xy}\right] + \frac{\partial}{\partial x}\left[\tau_{xy} + \omega_z \sigma_{xx}\right] = 0$$

1D Continuity equation



$$\frac{d}{dt}\int_0^L \frac{1}{1+\varepsilon_{x2}}\rho_o\, h\, w\, dx = \frac{V_1}{1+\varepsilon_{x1}}\rho_o\, h\, w - \frac{V_2}{1+\varepsilon_{x2}}\rho_o\, h\, w$$

$$\frac{d}{dt}\int_0^L (1-\varepsilon_{x2})dx = \frac{d}{dt}\left(-\varepsilon_{x2}L\right) = V_1\left(1-\varepsilon_{x1}\right) - V_2\left(1-\varepsilon_{x2}\right)$$

1D Continuity equation

2D continuity equation

$$\frac{d}{dt}\int_0^L \left[1-\left(\frac{\partial u}{\partial x}+\cancel{\frac{1}{2}\omega_z{}^2}\right)\right]dx = V_1\left(1-\varepsilon_{x1}\right)-V_2\left(1-\varepsilon_{x2}\right)$$

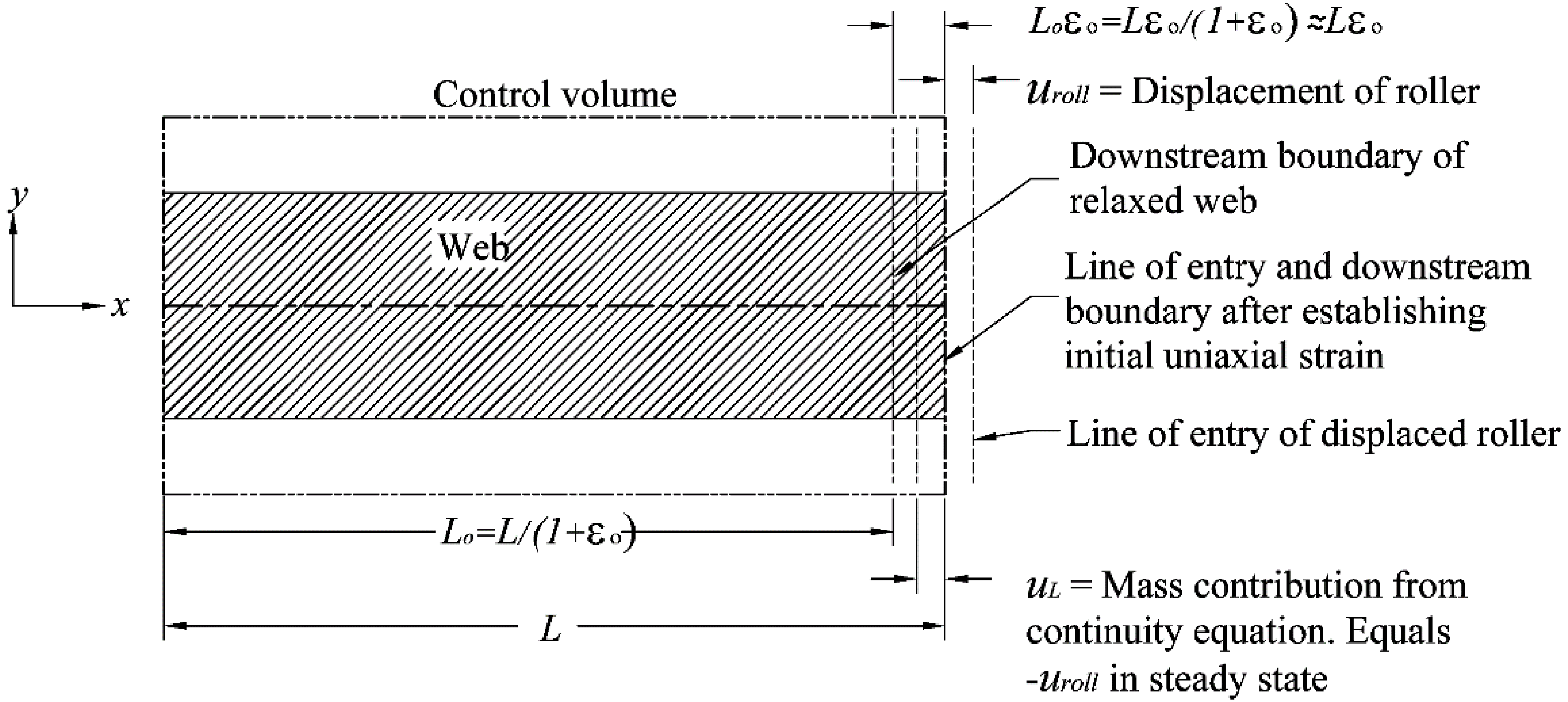
x-direction strain

Strains at x = 0 and x = L

$$\frac{\partial}{\partial t}\int_0^L \left[1-\frac{\partial u}{\partial x}\right]dx = \frac{\partial}{\partial t}\left(-u_L\right) = V_1\left(1-\varepsilon_0\right)-V_2\left(1-\varepsilon_{xL}\right)$$

x-direction
displacement
at x = L.

The control volume



The u displacement boundary condition

$$u_{boundary} = \underbrace{u_L}_{\text{Continuity equation}} + \underbrace{u_{roll}}_{\text{Roller motion}} + \underbrace{\varepsilon_o L}_{\text{Initial strain}}$$

Roller motion

$$u_{roll} = \underbrace{\theta_r}_{\text{Roller angle}} \left(y + \underbrace{v_L - z}_{\text{Lateral position of web relative to roller}} \right)$$

Roller angle

Lateral position of web relative to roller

$$\theta_r = \underbrace{\theta_{pivot}}_{\text{Maximum extent of roller pivot}} \underbrace{f_{ramp}}_{\text{Smooth cornered unit ramp}} \qquad z = \underbrace{d_{shift}}_{\text{Maximum extent of roller shift}} f_{ramp}$$

Maximum extent of roller pivot

Smooth cornered unit ramp

Maximum extent of roller shift

The v displacement boundary condition

$$v_{boundary} = \underbrace{v_L}_{\text{Normal entry}} - \underbrace{y\mu\varepsilon_o}_{\text{Initial strain}}$$

$$\frac{\partial v_L}{\partial t} = V_2\left[\theta_r - \underbrace{\frac{\partial v_L}{\partial x}\left(1+\frac{\partial u}{\partial x}\right)^{-1}}_{\text{Slope}}\right] + \frac{dz}{dt}$$

Normal entry equation

The complete model

$$\frac{\partial}{\partial x}\left[\sigma_{xx}-\omega_z\tau_{xy}\right]+\frac{\partial}{\partial y}\left[\tau_{xy}-\omega_z\sigma_{yy}\right]=0$$

x equilibrium

$$\frac{\partial}{\partial y}\left[\sigma_{yy}+\omega_z\tau_{xy}\right]+\frac{\partial}{\partial x}\left[\tau_{xy}+\omega_z\sigma_{xx}\right]=0$$

y equilibrium

$$\frac{\partial v_L}{\partial t}=V_2\left[\theta_r-\frac{\partial v_L}{\partial x}\left(1+\frac{\partial u}{\partial x}\right)^{-1}\right]+\frac{dz}{dt}$$

Normal entry

$$\theta_r=\theta_{pivot}\,f_{ramp} \qquad z=d_{shift}\,f_{ramp}$$

$$u_{roll}(y,t)=\theta_r\left(y+v_L-z\right)$$

$$\frac{\partial}{\partial t}\left(-u_L\right)=V_1\left(1-\varepsilon_{x0}\right)-V_2\left(1-\varepsilon_{xL}\right)$$

Continuity

$$u_{boundary}=u_L+u_{roll}+\varepsilon_o L$$

u boundary value

$$v_{boundary}=v_L-y\mu\varepsilon_o$$

v boundary value

Portion of FlexPDE script

```
theta_r = ang*pos(0.1, 0)
d_shift = 0
z = d_shift*pos(0.1, 0)
dtz = d_shift*vel(0.1, 0)

initial values
 U = exo*x
 V =  -y*exo*nu

equations
U: dx(Sx - wz*Txy) + dy(Txy - wz*Sy)  = 0
V: dx(Txy + wz*Sx) + dy(Sy + wz*Txy) = 0
V_b: dt(V_b) =  Vo*( theta_r -  vx/(1+ex)) + dtz
U_b: dt(-U_b) =  Vu/(1+exo)  - Vd/(1+ux)

boundaries
 region 1
   start 'perimeter1' (0, W/2)  mesh_density = 10
     value(U) = 0
     value(V) =  -y*exo*nu
   line to (0, -W/2)  mesh_density = 1
     load(U) = 0
     load(V) = 0
   line to (L, -W/2) mesh_density = 30
     value(U) =  U_b - theta_r*(y+V-z) + exo*L
     value(V) = V_b  - y*exo*nu
   line to (L, W/2) mesh_density = 1
     load(U) = 0
     load(V) = 0
   line to close
```

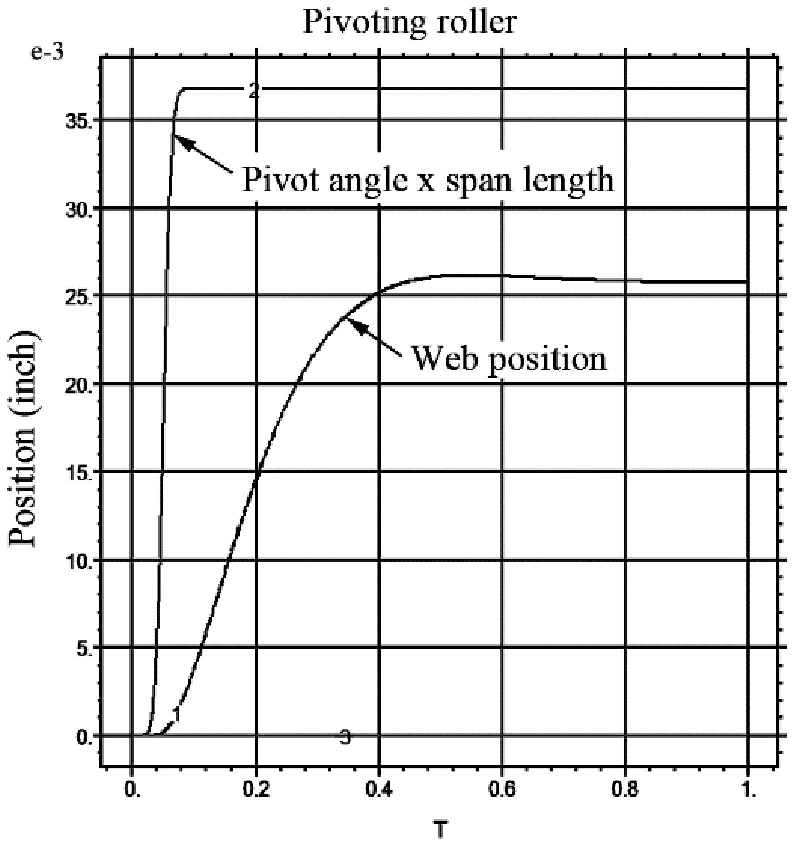
Comparisons with beam models

- The new model has not been tested experimentally; however, its behavior can be compared to the E-B static and dynamic models tested by Shelton in his dissertation. It can also be compared to a model that is closely related to Shelton's, but includes the effect of shear, (Timoshenko beam model) described in "The Effect of Mass Transfer on Multi-Span Lateral Dynamics of Uniform Webs"

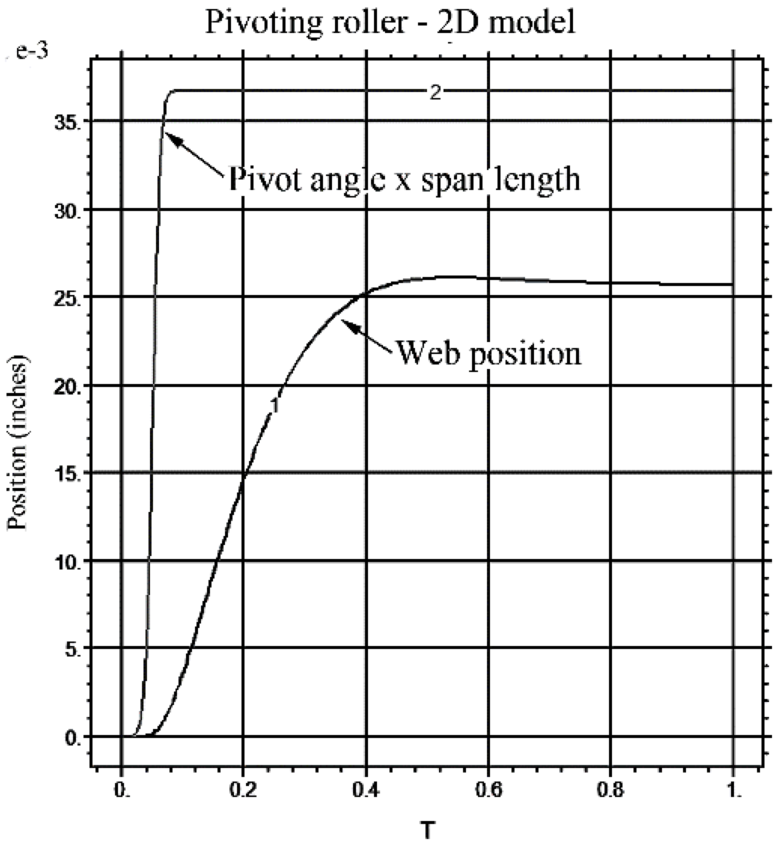
Comparisons with beam models

- The time histories in the next four figures compare outputs from the Timoshenko dynamic beam model with the 2D elasticity model for a roller pivot of 0.001885 radians. The input motion was a ramp function beginning at t = 0 with a rise time of 0.1 second.
- The application parameters are the same as Shelton’s first set of experimental parameters listed on page 45 of his dissertation. Tension = 36.7 Lbf, Span length = 19.5 inches, Width = 9.03 inches, thickness = 0.009 inch, KL = 0.2364, modulus = 450,000 psi. Line speed was 100 in/sec.

Lateral Position

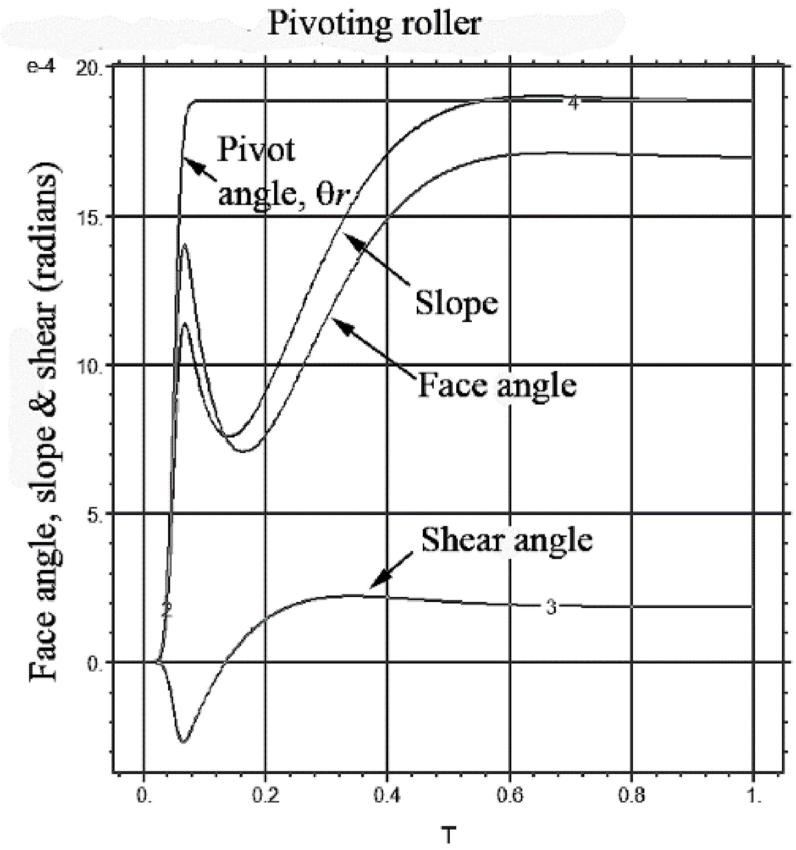


Beam

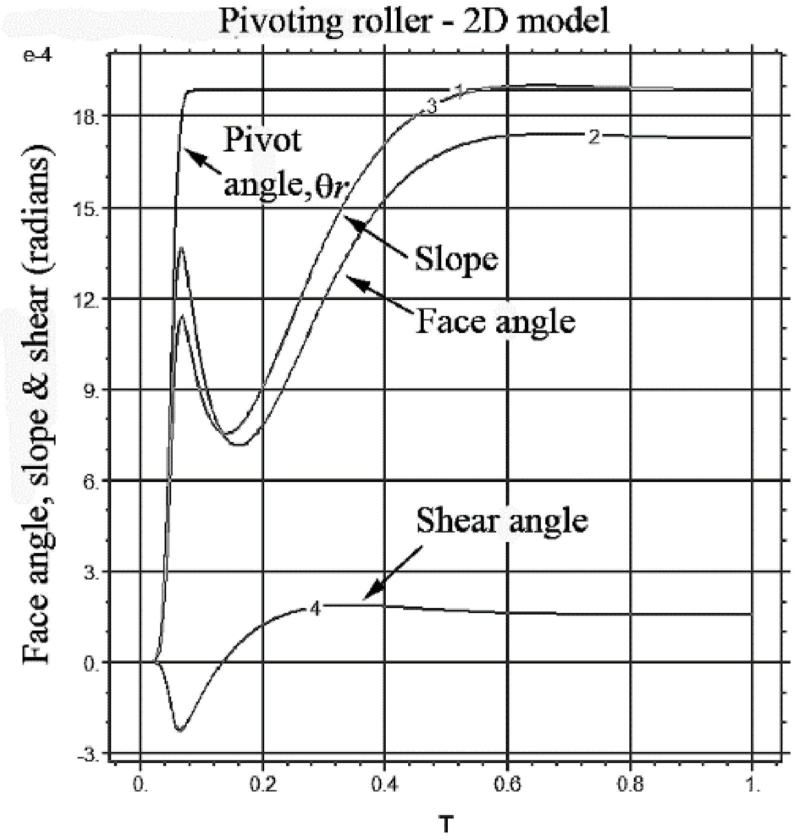


2D Elasticity

Face angle, slope & shear

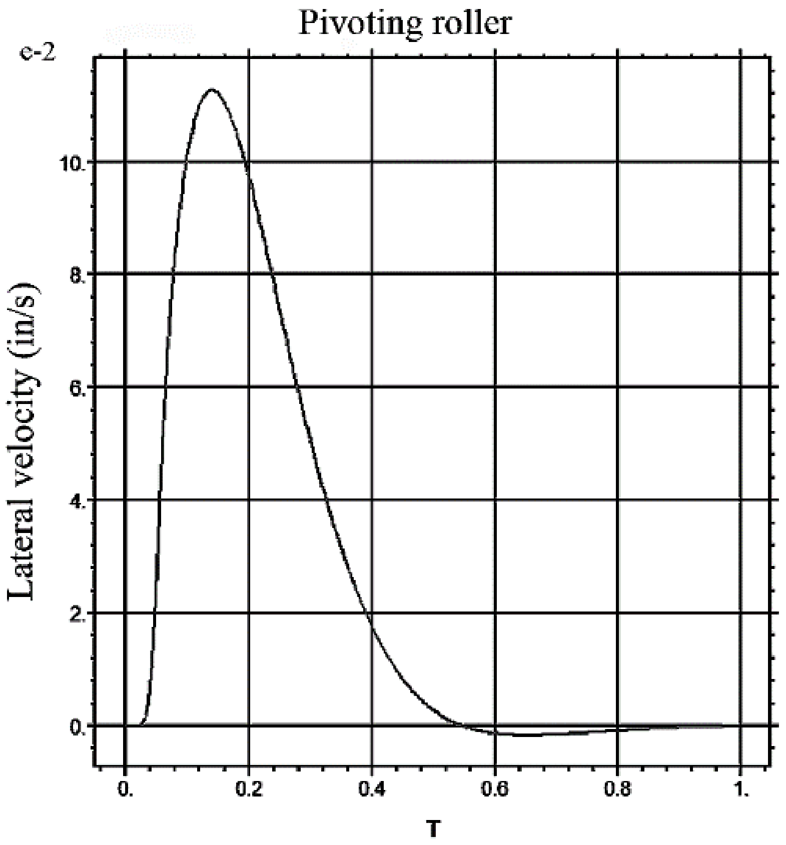


Beam

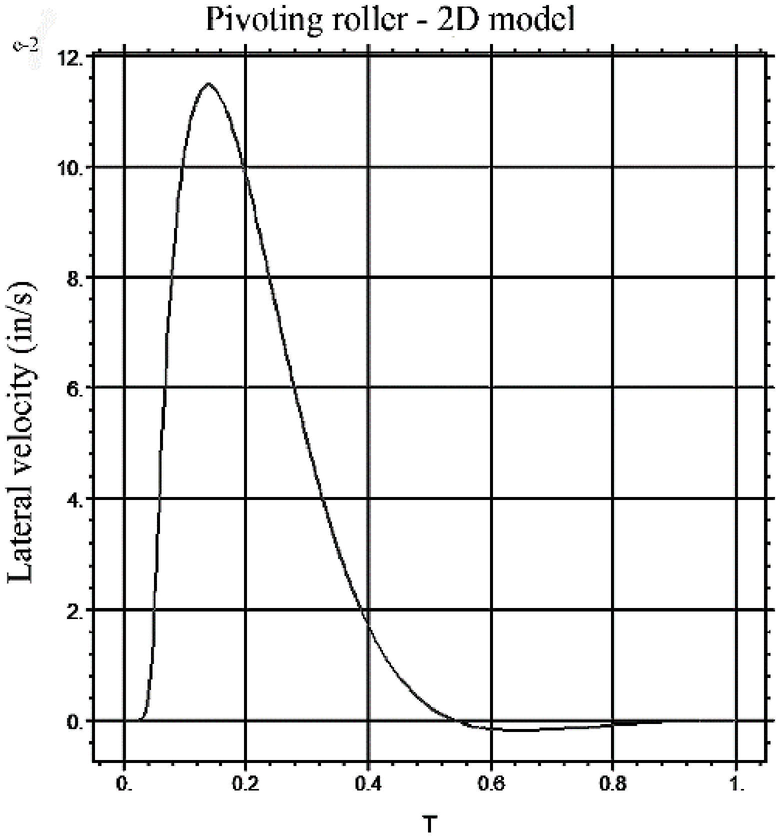


2D Elasticity

Lateral velocity

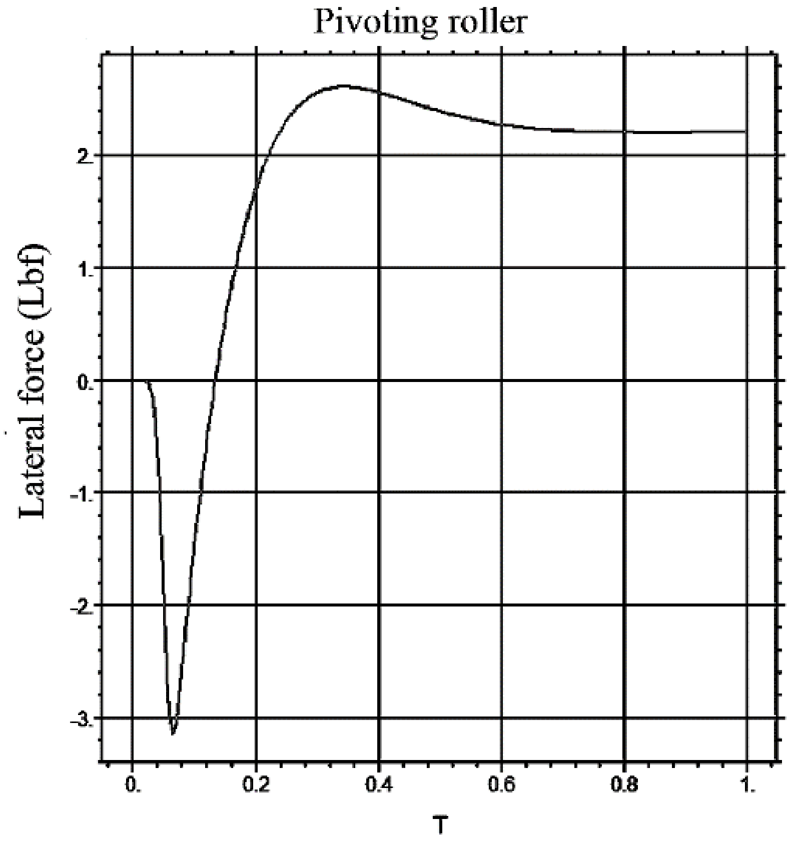


Beam

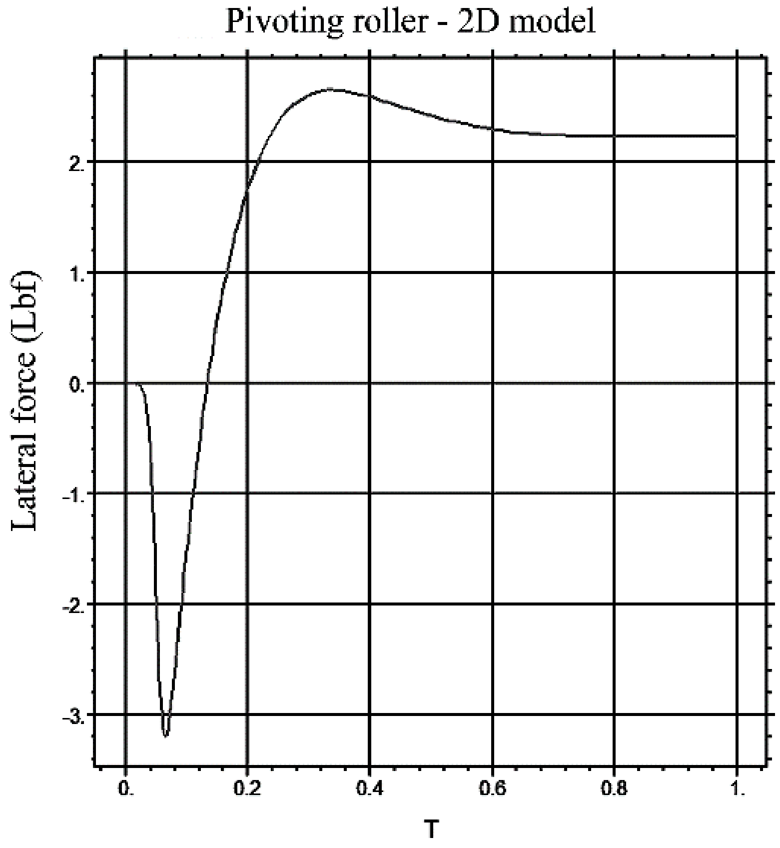


2D Elasticity

Lateral force

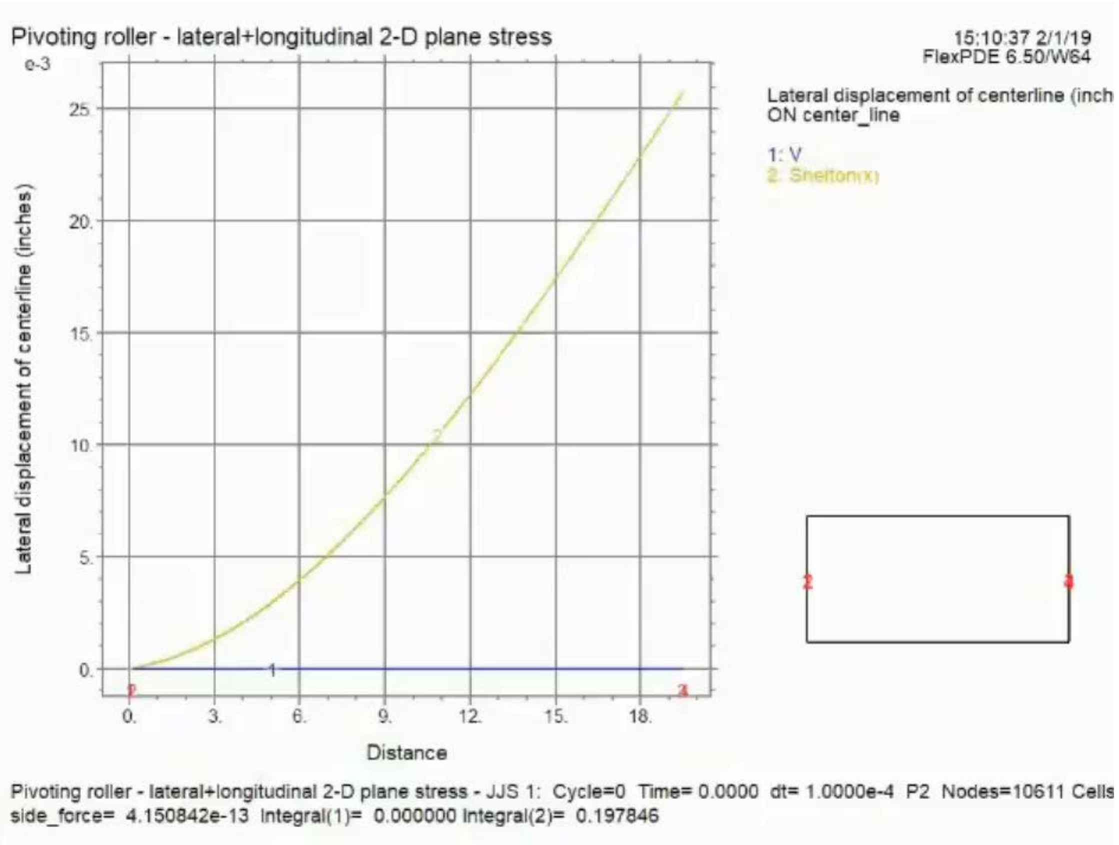


Beam



2D Elasticity

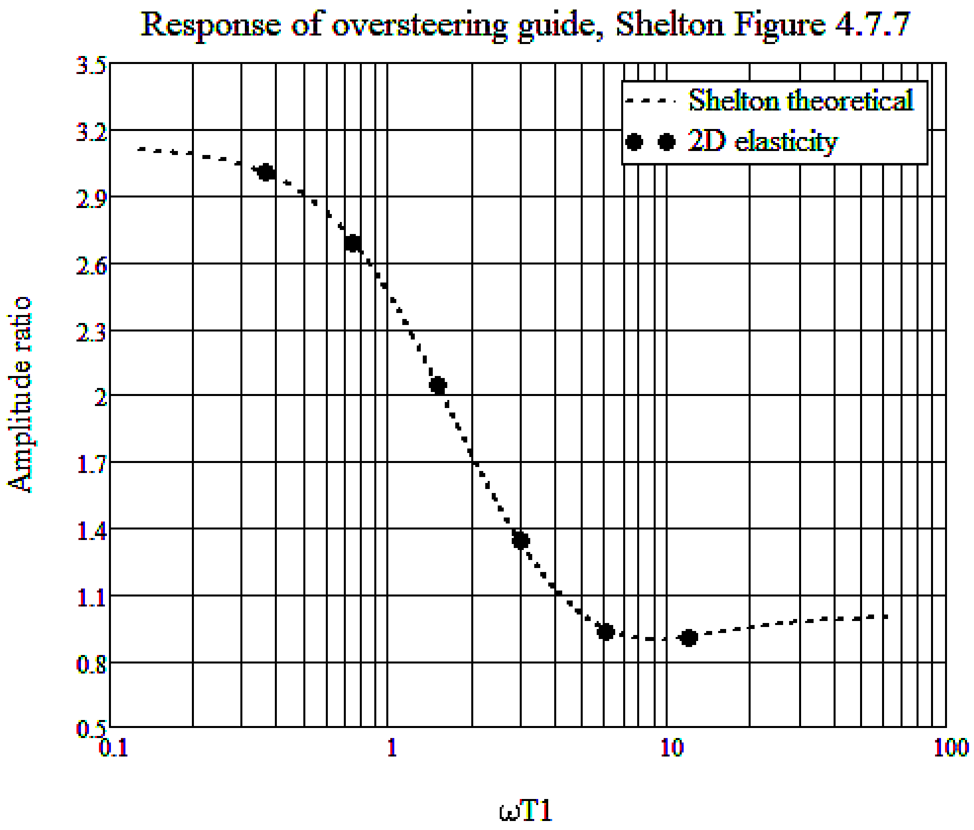
Comparison with Shelton steady state



Comparison with Shelton's dynamic model

- To verify his E-B dynamic model, Shelton ran four dynamic tests in which he applied a sinusoidal displacement to a downstream roller and measured the lateral displacement of the web. In two of them, a downstream roller was shifted on inclined linear bearings so that it simultaneously pivoted in the plane of the web about an instant center in the entering span (an arrangement commonly used in web guides).
- The dashed curve in the next graph shows the predicted amplitude response for one of them.
- Sinusoidal inputs were applied to the 2D elasticity model at the same frequencies used by Shelton for his tests and allowed to run for five time constants. The resulting amplitude ratios are plotted on the graph as black dots.

Comparison with Shelton's dynamic model



Parameters for Shelton's test are: Span length = 63 inches, Width = 1.5 inches, Thickness = 0.009 inches, Modulus = 510,000 psi, Tension = 30 Lbf, Speed = 100 inches/sec, Instant center radius = 18.09 inches.

Comparison with Shelton’s steady state test data

- Shelton tested his steady state E-B model by measuring lateral force at the downstream roller. Although the main purpose of the measurement was to validate the zero-moment boundary condition, he chose to use lateral force as a proxy because it was believed to be much easier.
- Comparison of results for all 21 of the tests are shown in the next slide.

Comparison with Shelton's steady state test

	Shelton Experimental data							SS E-B beam model		Dynamic Tim. Model		2D elasticity model	
#	1	2	3	4	5	6	7	8	9	10	11	12	13
	T (Lbf)	L (inch)	W (inch)	θ_r (rad)	N_L (Lbf)	KL	$N_L/T\theta_L$	N_L (Lbf)	% Diff	N_L (Lbf)	%Diff	N_L (Lbf)	% Diff
1	36.7	19.5	9.03	0.001885	2.375	0.2364	34.3	2.46	3.58	2.21	-6.95	2.24	-5.68
2	36.7	19.5	9.03	0.001885	2.45	0.2364	35.4	2.46	0.41	2.21	-9.80	2.24	-8.57
3	55.1	19.5	9.03	0.001885	2.4	0.2904	23.1	2.45	2.08	2.2	-8.33	2.23	-7.08
4	36.7	40	9.03	0.00377	1.1	0.485	7.95	1.15	4.55	1.12	1.82	1.13	2.73
5	55.1	40	9.03	0.00377	1.075	0.594	5.17	1.14	6.05	1.11	3.26	1.12	4.19
6	18.3	56.5	9.03	0.00377	0.575	0.484	8.34	0.576	0.17	0.568	-1.22	0.573	-0.35
7	55.1	56.5	9.03	0.00377	0.55	0.842	2.644	0.554	0.73	0.547	-0.55	0.552	0.36
8	55.1	63	4.48	0.01884	0.15	2.684	0.1445	0.164	9.33	0.164	9.33	0.165	10.00
9	9.1	40	4.48	0.00377	0.125	0.694	3.64	0.138	10.40	0.137	9.60	0.137	9.60
10	36.7	40	4.48	0.00377	0.125	1.392	0.903	0.122	-2.40	0.122	-2.40	0.123	-1.60
11	36.7	40	4.48	0.00941	0.325	1.392	0.941	0.305	-6.15	0.304	-6.46	0.306	-5.85
12	36.7	20	4.48	0.00377	0.525	0.696	3.79	0.55	4.76	0.537	2.29	0.542	3.24
13	9.1	20	4.48	0.00377	0.55	0.346	16	0.567	3.09	0.553	0.55	0.557	1.27
14	18.3	20	4.48	0.00377	0.575	0.491	8.34	0.561	-2.43	0.547	-4.87	0.552	-4.00
15	27.5	20	4.48	0.00377	0.625	0.601	6.03	0.556	-11.04	0.542	-13.28	0.547	-12.48
16	36.7	20	4.48	0.00377	0.625	0.696	4.51	0.55	-12.00	0.537	-14.08	0.542	-13.28
17	9.1	40	4.48	0.00377	0.125	0.694	3.64	0.138	10.40	0.137	9.60	0.138	10.40
18	36.7	40	4.48	0.00377	0.125	1.392	0.904	0.122	-2.40	0.122	-2.40	0.123	-1.60
19	36.7	40	4.48	0.00941	0.325	1.392	0.941	0.305	-6.15	0.304	-6.46	0.306	-5.85
20	36.7	56.5	4.48	0.01884	0.25	1.967	0.362	0.263	5.20	0.262	4.80	0.264	5.60
21	55.1	56.5	4.48	0.01884	0.2	2.408	0.1925	0.226	13.00	0.227	13.50	0.228	14.00

5 on

8 on

10 on

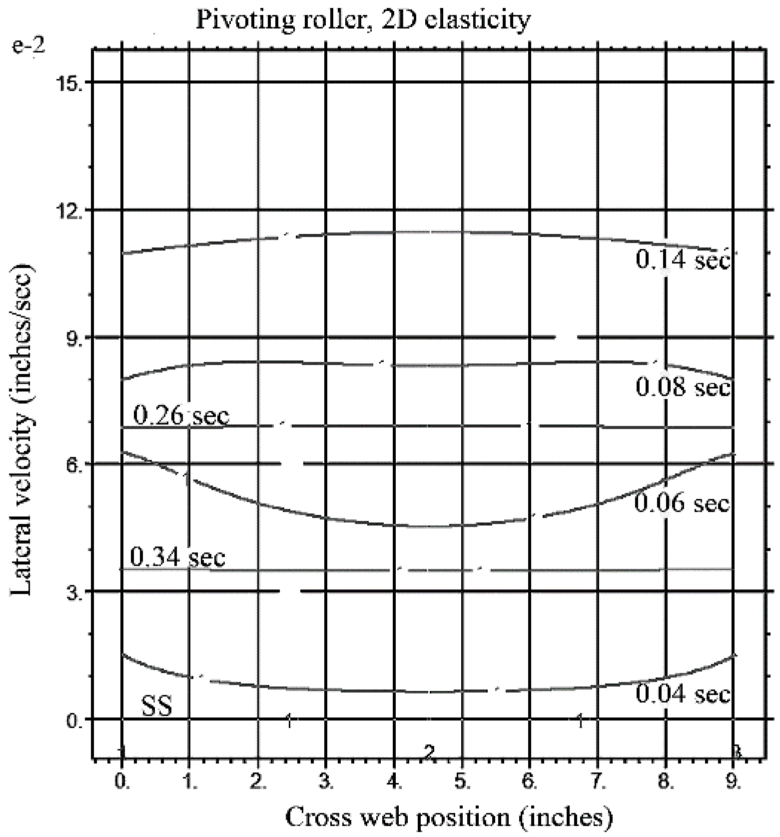
12 on

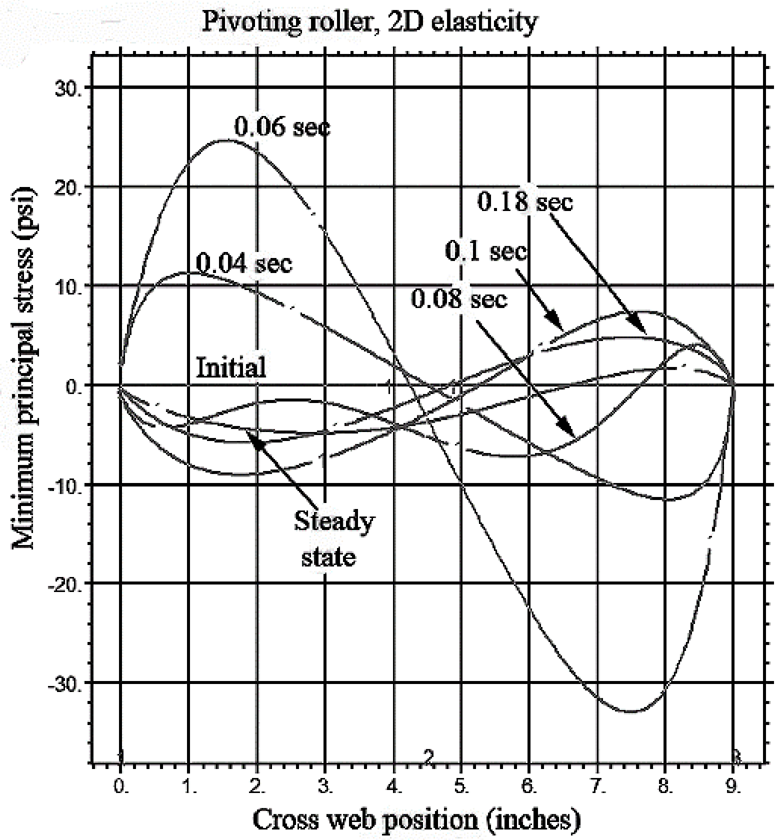
Model predictions for L/W<2

Parameters						SS E-B beam model	Dynamic Tim. Model	2D elasticity model
1	2	3	4	5	6	7	8	9
L/W	T (Lbf)	L (inch)	W (inch)	θ_r (rad)	θ_{cr} (rad)	N_L (Lbf)	N_L (Lbf)	N_L (Lbf)
2.16	36.7	19.5	9.03	0.001885	0.00215	2.46	2.21	2.24
1.62	36.7	14.625	9.03	0.0014138	0.00162	3.28	2.74	2.79
1.08	36.7	9.75	9.03	0.0009425	0.00108	4.93	3.41	3.49
0.54	36.7	4.875	9.03	0.0004713	0.000541	9.87	3.54	3.64

It is apparent that for values of L/W below 2.0, the Timoshenko and elasticity models are in fair agreement, but the E-B model diverges significantly from the other two. Could this mean that the Timoshenko model can be used for small L/W? This is an area that requires future testing.

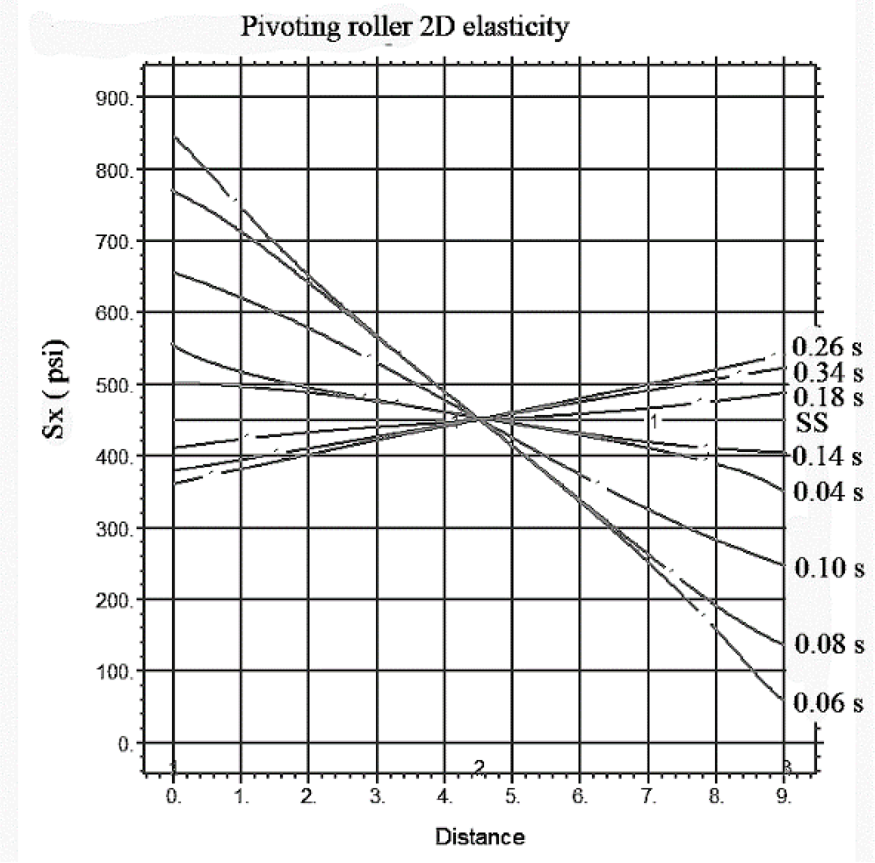
Dynamic wrinkling



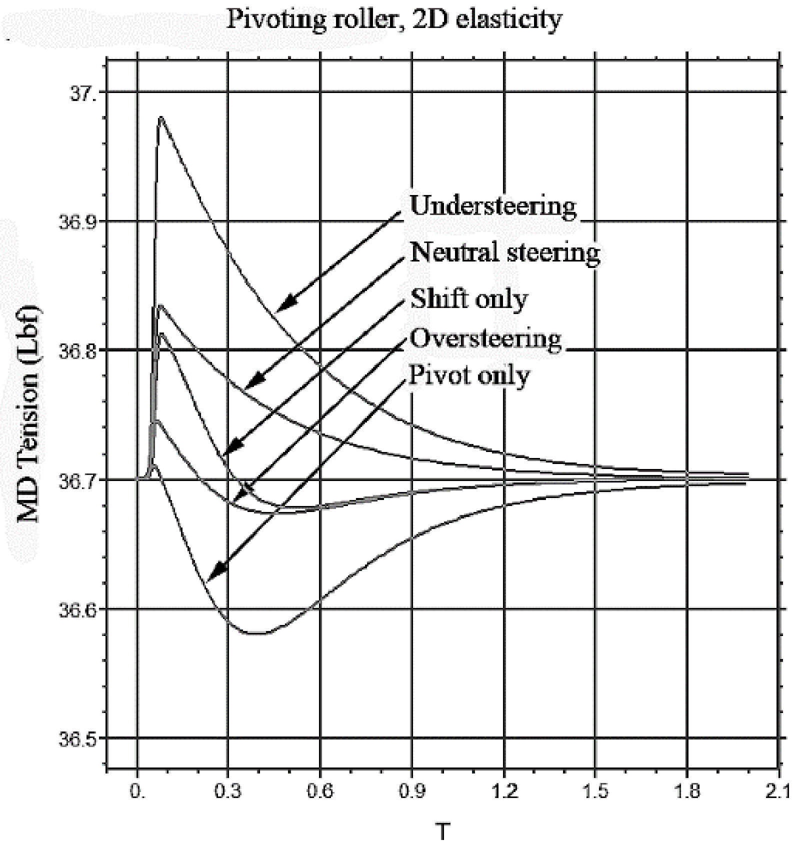


When it’s moving laterally, the web velocity varies with position.
This has implications for wrinkling because the faster portions will advance on the portions ahead of them and create lateral compressive stress.

MD stress for pivoting roller



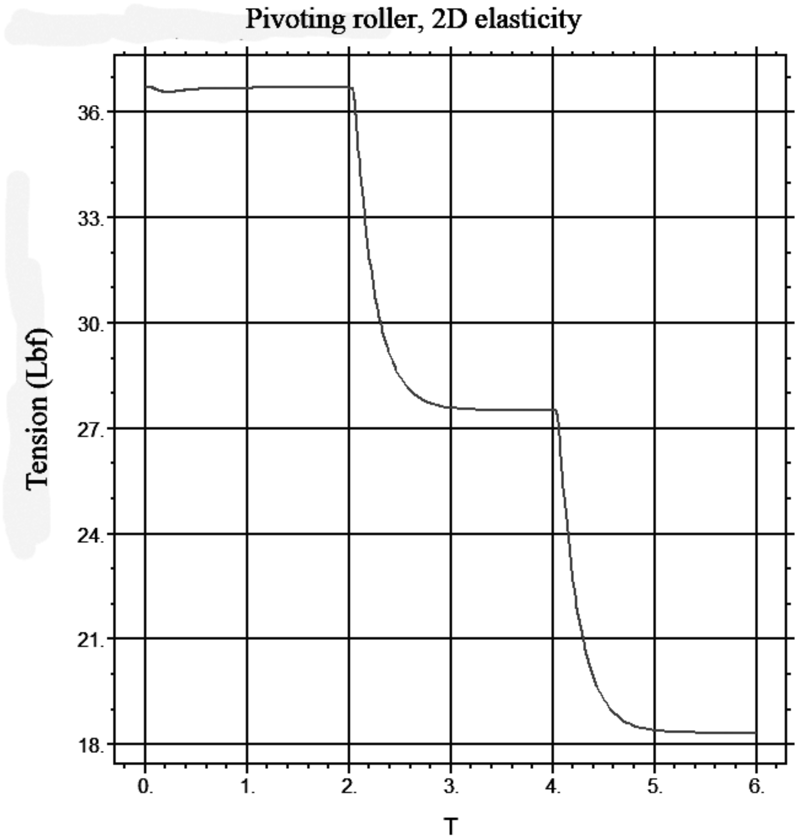
Tension disturbances due to roller motion



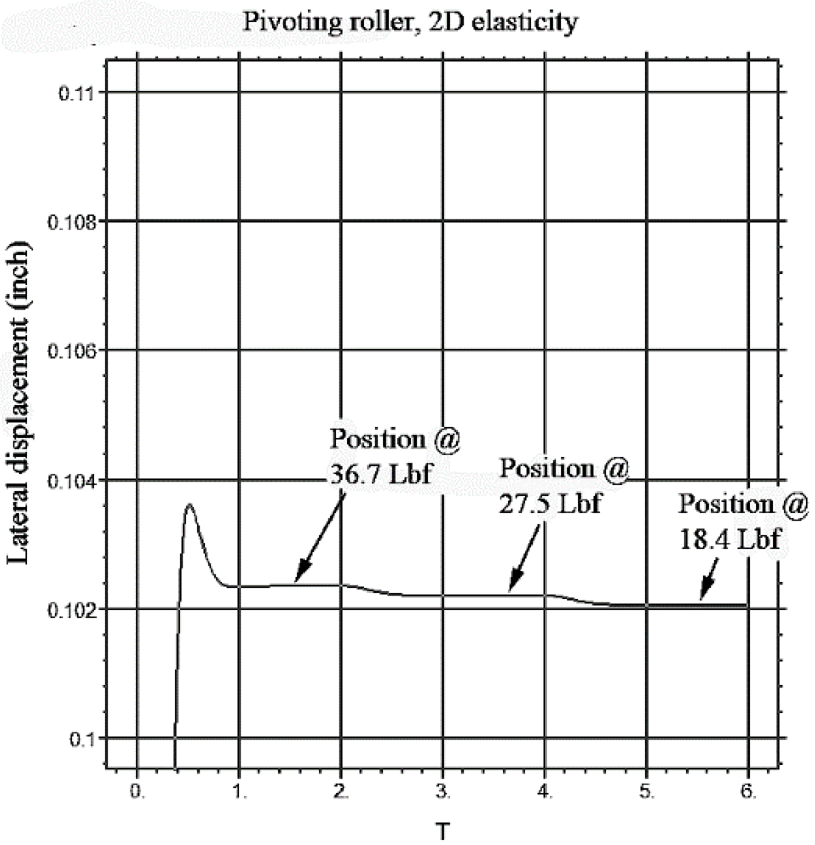
Application parameters correspond to Shelton's fourth set of experimental parameters listed on page 45 of his dissertation [2]. Tension = 36.7 Lbf, Span length = 40 inches, Width = 9.03 inches, thickness = 0.009 inch, KL = 0.485, modulus = 450,000 psi.

1. Pivot-only – Pivot angle = 0.00377 radians, lateral shift = 0.
2. Oversteering – Pivot angle = 0.00377 radians, lateral shift = 0.053 inch
3. Shift-only – Pivot angle = 0, lateral shift = 0.105 inch
4. Neutral steering – Pivot angle = 0.00377 radians, lateral shift = 0.105 inch
5. Understeering – Pivot angle = 0.00377 radians, lateral shift = 0.158 inch

Effect of tension change on lateral position



Tension



Lateral position

Conclusions

- The model performed well in the following comparisons.
  - Timoshenko beam model, of a pivoted roller
    - Lateral position
    - Face angle, Slope and Shear
    - Lateral velocity
    - Lateral force
  - Shelton’s steady state E-B model - lateral displacement vs distance along span
  - Shelton’s dynamic E-B model - frequency response of an oversteering guide
  - Shelton’s 21 tests of steady state lateral force

Conclusions

- Although new testing should be done, particularly for $L/W < 2$, the remarkably close agreement with Shelton's static and dynamic test results gives the 2D elasticity model a high degree of credibility and is suggestive of a new conceptual context for further study of lateral web dynamics.
- Examples illustrated in this paper were chosen mostly to enable comparison with tested configurations. Implications of the new model, such as tension interaction and dynamic CD behavior, should be explored in areas of application such as,
  - Typical web guide configurations
  - Nonuniform rollers
  - Nonuniform webs

Conclusions

- An important limitation of the new model is that it cannot be incorporated directly into control algorithms. Its main utility for control engineers will be its ability to identify and quantify the interaction of lateral and longitudinal systems.