A NEW METHOD FOR ANALYZING THE DEFORMATION AND LATERAL TRANSLATION OF A MOVING WEB

by

J. L. BROWN
ESSEX SYSTEMS
USA

ABSTRACT

A new method is presented for modeling the elastic behavior of webs conveyed over rollers. In addition to producing detailed descriptions of stress fields, it provides a new way of looking at problems that will help web process engineers form a better physical picture of web behavior.

The new method is based on two fundamental boundary conditions that define web behavior at the point of entry of the web onto a downstream roller. One is a generalization of an existing geometric concept called the ***normal entry rule***. The other, presented here for the first time, is based on application of the principle of conservation of mass. For reasons that will become apparent, it is called the ***normal strain rule***. This paper will show that these two rules, together with a nonlinear version of the equations for two-dimensional plane stress enable the solution of a large class of unsolved web handling problems. Numerical solutions are calculated with a finite-element partial differential equation solver.

The basic principles are developed and shown to produce results that are in agreement with published test results for two problems that have been solved by other means.

1. Lateral displacement at a misaligned roller: Results are compared to Shelton's 1968 thesis [2].
2. Lateral displacement at a tapered roller: Results are compared to the 2001 IWEB paper by Markum and Good [4].

In both cases, the referenced studies used beam theory to predict lateral displacement. The experimental results are compared with predictions from the new method (P. D. E. model) and shown to be in excellent agreement.

Following a brief summary of the background of the problem, the new boundary condition is derived and other relevant equations summarized. Results are then computed and compared with results for the two cases mentioned above.

NOMENCLATURE

D	Roller diameter, m
G	Shear modulus, Pa
h	Web thickness, m
L	Length of span, m
Q_i	Mass flow rate per unit of relaxed area at upstream roller, Kg/s
Q_o	Mass flow rate per unit of relaxed area at downstream roller, Kg/s
u	Particle displacement in *x* direction, m
u_y	Derivative of *u* with respect to *y*
v	Particle displacement in *y* direction, m
v_x	Derivative of *v* with respect to *x*
V_y	Velocity in the y direction, m/s
V_s	Velocity of web along axis of roller, m/s
V_u	Surface velocity of upstream roller, m/s
V_d	Surface velocity of downstream roller, m/s
Y	Lateral displacement at downstream roller
γ_{xy}	Elastic shear strain
ε	Elastic strain
ε_o	Longitudinal strain at entry of upstream roller
η	Deformed *y* coordinate, m
θ_n	Angle of path of web particles relative to normal to roller axis (CCW positive), radians
θ_r	Angle of axis of misaligned roller relative to y-axis (counterclockwise positive), radians
μ	Poisson's ratio
ξ	Deformed *x* coordinate, m
ρ_o	Mass density of relaxed web, Kg/m^3
ρ_u	Mass density of stressed web at upstream roller, Kg/m^3
ρ_d	Mass density of stressed web at downstream roller, Kg/m^3
σ	Stress, Pa
σ_n	Stress normal to a boundary, Pa
τ_{xy}	Shear stress in *x*,*y* plane, Pa
ψ	Angle of tangent to particle trajectory of web (in relation to x-axis), radians
ω_z	Elastic rotation in *x*,*y* plane

Subscripts

u	Upstream
d	Downstream
x	Aligned with x-axis
y	Aligned with y-axis
z	Aligned with z-axis (normal to web plane)

INTRODUCTION

Beam theory has been applied very successfully to many problems in web handling. Some, however, have resisted solution because they require the specification of unknown forces or moments. Furthermore, beam theory is limited in its ability to provide insight into details of deformations and displacements of the web, particularly at the web-roller interface. These deformations may be caused by non-uniformities such as,

1. Variations in the roller diameter along its axis (Tapered, crowned and reverse-crowned rollers).
2. Curvature in the roller axis (bowed roller)
3. Cambered webs (Curvature along the longitudinal axis of the relaxed web).

Beam theory models have many virtues. They provide closed form solutions. They permit quick estimates in a familiar engineering context. And they provide insight into the relationships of key parameters. However, they have two problems. First, in cases such as the cambered web, it has been difficult to define satisfactory boundary conditions. Beam theory requires four conditions. Two for the upstream end are usually easy and the normal entry rule provides one of the two needed for the downstream end. But, there is always difficulty in finding the fourth condition. Second, beam theory is poorly suited to addressing problems where details of the cross web deformation are desired. The approach taken in this paper is to look at the problem from the standpoint of elasticity theory.

Two-dimensional elasticity theory for web spans requires two boundary conditions at each edge. All of them are straightforward except for the two at the downstream roller. The normal entry rule is one, provided it is generalized for use in the context of elasticity theory. The second, which will be called the normal strain rule, is developed and presented here for the first time. It is based on the principle of the conservation of mass and has the same range of application as the normal entry rule. This paper will show how these concepts can be combined with a non-linear form of the two-dimensional equations of plane stress to solve many web handling problems, including those mentioned above.

An important difference between the new method (P. D. E. model) and beam theory is that the boundary conditions at the rollers are expressed entirely in terms of displacements and strains. This requires a change in perspective from beam theory. Deformation is viewed as a cause rather than a consequence of the web's behavior.

A general-purpose finite-element partial differential equation solver is used to generate detailed solutions for stresses and lateral translations throughout spans. The P. D. E. solver software for this work, running on an ordinary 2.8 GHz PC, produced solutions to problems in times ranging from 1 to 10 minutes.

ASSUMPTIONS

For this analysis, air film effects are ignored and it will be assumed that friction controls traction between the roller and web. The usual assumptions are made about the behavior of the web when it is on the roller. At the entry point, friction locks it onto the roller surface in a *stick zone*. Any strains existing at the point of entry are then frozen in place and remain fixed relative to the roller surface until the web reaches a zone at the exit where it begins to slip from the roller under the influence of stresses downstream. Furthermore, the lines of contact at entrance and exit are assumed to be parallel to the roller axis. The turning torque of the roller is usually small in comparison to web tension and the *stick zone* will be a large percentage of the contact area. Under these conditions, the stresses in the upstream span are isolated from changes downstream.

Other assumptions: Viscoelastic and inertial effects are not significant. Thickness and material properties are constant in the longitudinal direction.

THE NORMAL ENTRY RULE

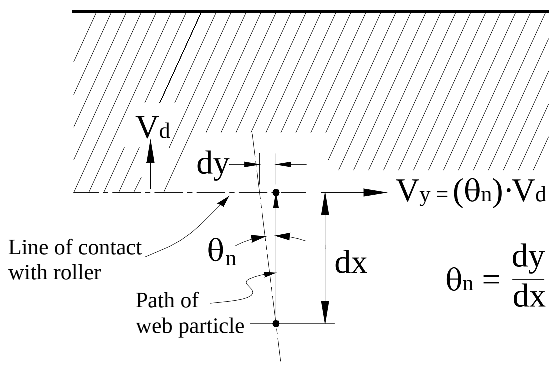

Figure 1
Normal entry rule

Current beam models assume that a web acts as an ensemble of fibers whose orientation matches the neutral axis of the beam. The normal entry rule, illustrated in Figure 1, then states that as a web moves onto a roller, it moves laterally at a rate proportional to the tangent of the angle between the neutral axis and the normal to the roller axis. Furthermore, the direction of the lateral motion takes the web toward a steady state condition in which the neutral axis is normal to the roller axis. This relationship is expressed in equation (1),

$$V_y = \tan(\theta_n) \cdot V_d \approx \theta_n \cdot V_d \tag{1}$$

where V_y is the web velocity along the roller axis, θ_n is the angle of the neutral axis and V_r is the roller surface velocity. In the steady state, θ_n becomes zero. It departs from zero when there is a disturbance in the roller alignment or the lateral position upstream. In practice, θ_n is almost always small enough to use the approximation on the right. For uniform webs with good traction on uniform rollers, the validity of the normal entry rule was demonstrated by Shelton [2] in his dissertation.

It is intuitively evident that uniform deformation cannot be assumed for all points across the web, especially in the case of non-uniform webs or rollers. It is, thus, unclear how θ_n should be defined for these cases. Later in this paper, a methodology will be developed that removes this difficulty.

MASS FLOW

There are many circumstances when the normal entry rule is not sufficient, by itself, to explain lateral web behavior. One such instance is the spreading action of a reverse-crown (concave) roller illustrated in **Figure 2**. It is an established fact that such rollers have a spreading action. Swanson [1] describes a very convincing demonstration. Spreading can even be produced on a straight roller by wrapping a thin narrow band of tape around the circumference of a roller just inside the web edges to produce a “stepped” concave roller.

Experienced practitioners of web handling explain the spreading as follows. At the edges of the concave roller, material is transported faster from the span than at the center because the circumferential velocity is higher there (V1 > V2). This alters the stress distribution across the span in such a way that the normal entry rule can be satisfied only if the web spreads laterally.

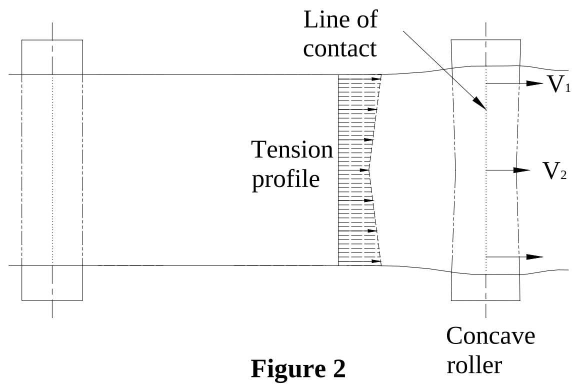

Figure 2
Concave Roller

Although this concept has been used for many years to provide an intuitive explanation of wrinkling and spreading (Notably, by Feiertag in the Web Handling Center’s semiannual Web Handling Seminar) no one has, so far, found a way to incorporate the concept into a comprehensive quantitative model.

EARLIER WORK

Shelton [2], (Pg. 29) recognized the role of conservation of mass in his 1968 dissertation when he derived the steady-state elastic curve of a web moving onto a misaligned roller. One of the boundary conditions required for the solution is that the moment acting on the end of the span at the line of contact between it and the downstream roller is zero. He determined this condition through experimentation and a process of elimination. At the end of his derivation, he pointed out that the normal entry rule had been confirmed by his experiments and then he gave an intuitive proof of the zero moment condition that was essentially equivalent to stating that the mass flow of a uniform web, at all points across the line of contact with a misaligned uniform roller, had to be constant. It is clear from this that Shelton saw a connection between mass flow and the zero moment boundary condition.

Swanson [3] in his 1997 paper on web spreading devices recognized that mass flow and the normal entry rule interact to produce spreading on a concave roller. To prove that the roller actually spread the web, he positioned a slitting blade immediately before the span he was observing. He used this to separate an inch of the web (0.8 mil PET) at the edge and measured the displacement of this piece at the downstream roller. To explain the observed displacement, he made calculations using a beam model. The calculations show that he recognized the importance of the cross-web tension profile. But, there was no recognition in the paper of the possibility of using mass flow in the way that is proposed here.

Markum and Good [4] developed a beam model that could provide an indication of the spreading effect of a contoured roller by slitting the web at the upstream roller in a manner similar to Swanson [1]. Since only the separation of the two halves was calculated, this was actually an analysis of web behavior on a tapered roller and provided no insight into the details of stress distribution of an unslit web along the roller axis. In their analysis they recognized a connection between mass flow rate and longitudinal strain and then used it to develop an estimate of end moment for a beam model. But, as with Swanson [1], they treated the relationship as an approximation. Experiments were made that showed their model made estimates of lateral displacements that were in the range of 5 to 20% of measured results. Data from these tests can be used as a test of the model proposed here.

DEFINITIONS AND A FRAME OF REFERENCE

The first step in analyzing the problems described in the introduction is to establish terminology and a frame of reference.

First the terminology for plane stress will must be defined. This will include a two-dimensional version of Novoshilov's nonlinear equations of equilibrium for small rotations.

Two other issues also require attention. First, there is the fact that elasticity theory usually treats static problems. But, the web is moving. Second, elasticity theory usually treats problems in which the boundaries are specified for the relaxed state. But, the upstream and downstream boundaries of a web are specified for the deformed state.

Plane Stress Definitions

The following equations for plane stress are taken from Novoshilov's [5] simplified nonlinear theory for small rotations.

Displacements from the relaxed coordinates *x* and *y* are *u* and *v*, respectively. Strains are defined as follows.

Strain in the *x* direction $$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\omega_z^2 \cong \frac{\partial u}{\partial x} \qquad (2)$$

Strain in the *y* direction $$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\omega_z^2 \cong \frac{\partial v}{\partial y} \qquad (3)$$

Shear strain $$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (4)$$

$$u_y = \frac{\partial u}{\partial y} \qquad (5) \qquad\qquad v_x = \frac{\partial v}{\partial x} \qquad (6)$$

Strain in the *z* direction $$\varepsilon_z = \frac{-\mu}{1-\mu}(\varepsilon_x + \varepsilon_y) \qquad (7)$$

Rotation in *x*,*y* plane $$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (8)$$

Deformed coordinates are

$$\xi = x + u \qquad (9) \qquad\qquad \eta = y + v \qquad (10)$$

Assuming Hook's Law, the stresses may be expressed in terms of strains, Poisson's ratio, μ, and modulus of elasticity, *E*, as follows.

The x-axis stress is: $$\sigma_x = \frac{E}{1-\mu^2}\left[\varepsilon_x + \mu\varepsilon_y\right] \tag{11}$$

The y-axis stress is: $$\sigma_y = \frac{E}{1-\mu^2}\left[\varepsilon_y + \mu\varepsilon_x\right] \tag{12}$$

The shear stress is: $$\tau_{xy} = \frac{E}{2(1+\mu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \tag{13}$$

The equations of equilibrium are:

$$\frac{\partial}{\partial x}\left[\sigma_x - \omega_z\tau_{xy}\right] + \frac{\partial}{\partial y}\left[\tau_{xy} - \omega_z\sigma_y\right] = 0 \tag{14}$$

$$\frac{\partial}{\partial y}\left[\sigma_y + \omega_z\tau_{xy}\right] + \frac{\partial}{\partial x}\left[\tau_{xy} + \omega_z\sigma_x\right] = 0 \tag{15}$$

Classical linear elasticity theory is based on the assumption that rotation ω_z is so small that its effect is negligible. This is not true in web handling problems where both curvature and longitudinal tension are significant. Without $\omega_z\sigma_x$ in the equation of equilibrium (15) the effects of longitudinal tension on the elastic curve will not be reflected in the results. Its net effect in web analysis is very similar to the second term in Shelton's differential equation for the elastic curve of a web under tension at a misaligned roller[2], (Pg. 58).

The effect of rotation may be safely ignored in the definitions of ε_x and ε_y in equations (2) and (3), because it appears by itself to the second power and it is of the same order of magnitude as the elongations and shears.

The transformations from undeformed to deformed coordinates are

$$d\xi = \left(1+\varepsilon_x\right)dx + u_y dy \qquad d\eta = v_x dx + \left(1+\varepsilon_y\right)dy \tag{16}$$

Web Motion

In the steady-state, the shape of a deformed web span is stationary, even though material is constantly moving through it. If a series of marks are made across the web, they will follow curved paths as they move down the span. By definition, the existence of a steady-steady state means that another series of dots, made at the same positions at a later time, will follow the same curves. These curves are, therefore, the trajectories of individual particles, similar to streamlines in a flowing fluid. Observations show that so long as traction is maintained on the rollers, the curved trajectories are independent of web velocity (assuming no inertial effects). Therefore, the web can, in principle, be brought to a complete stop without altering its shape and be analyzed as though it is a fixed, elastic sheet with boundary conditions that were established while it was moving.

Boundaries and Orientation of the Relaxed Web

First, consider a uniform web that is moving under tension over a series of perfectly aligned rollers, as illustrated in Figure 3. Under these conditions a web will be observed to have uniform width and zero stress in the y direction. The width is reduced by Poisson's ratio and there is no "neck down" region near the end points of the span. It will be seen later that this is consistent with a rigorous definition of the normal entry rule. Thus, the web is in pure tension.

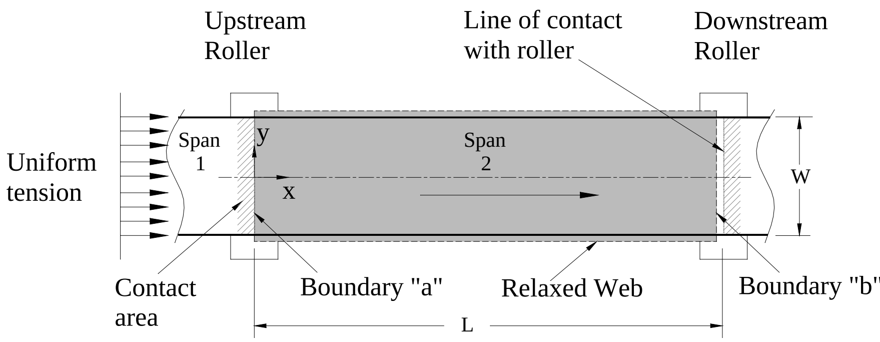

Figure 3
Reference geometry for a simple case

If it is now cut along the lines of contact with the downstream and upstream rollers, the web will relax into a rectangular sheet. The length will be reduced and there will be a small width increase due to Poisson's ratio. When the web is under tension, deformation at boundary "a" is determined by conditions in span 1. Therefore, when the effects of changes in boundary conditions at "b" are studied, the web may be modeled as though it is anchored at "a" with zero *x* displacement. The *y* displacement will be due solely to conditions at the entry of the upstream roller. If span 1 is in pure longitudinal tension, this will be due only to Poisson's ratio. Furthermore, changes in span 2 will not affect span 1 because of the isolation provided by traction on the upstream roller. So, at boundary "a",

$$u = 0 \quad (17) \qquad\qquad v = -\mu\varepsilon_o y \ . \quad (18)$$

Therefore, assuming the tension in span 2 is ε_{xo}, the relaxed dimensions are,

$$L\left(1-\varepsilon_o\right) \quad by \quad W\left(1+\mu\varepsilon_o\right) \ . \quad (19)$$

And since the mass flow at each point along boundary "a" is the same, the longitudinal axis of the relaxed web will be aligned with the x-axis.

Next, consider what happens when conditions are changed at boundary "b". Take, for example, the case of a misaligned roller, illustrated in Figure 4. It is known that at a misaligned roller the web will advance laterally until it reaches a steady state position at which its direction of motion is normal to the roller axis. A particle starting at boundary "a" will follow a curved trajectory. And since the boundary conditions at "a" are exactly the same as the simple case of Figure 3, the longitudinal axis of the relaxed web will again be aligned with the x-axis. And all particles that follow curved trajectories in the deformed web will follow straight lines in the relaxed web. Now that the frame of reference has been established, it is possible to state clearly what the normal entry rule means in the context of the theory of elasticity.

<u>Boundary Defect</u>

If a web that is running onto a roller under tension could be stopped and then cut along the line of entry, the cut edge of the relaxed web would not match the original boundary. In most cases, this effect is so small that it can be ignored. But, for very short spans involving nonuniform rollers it can become a significant fraction of the average downstream displacement. One way to deal with it is to solve a problem recursively. First it is solved with the relaxed boundary the same as line of entry. Then, information from the solution is used to adjust the boundary so that it will be match the line of entry after deformation. This can be done efficiently with a P. D. E. solver.

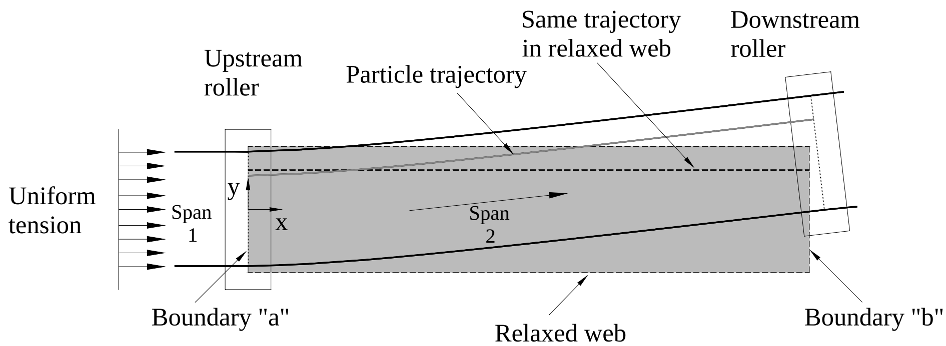

Figure 4
Reference geometry

THE NORMAL ENTRY RULE IN THE CONTEXT OF ELASTICITY THEORY

The normal entry angle must be redefined so that it can be used at each point across the width of an elastically deformed web. It is apparent that something more is needed than "orientation of the neutral axis" or "average orientation of longitudinal fibers".

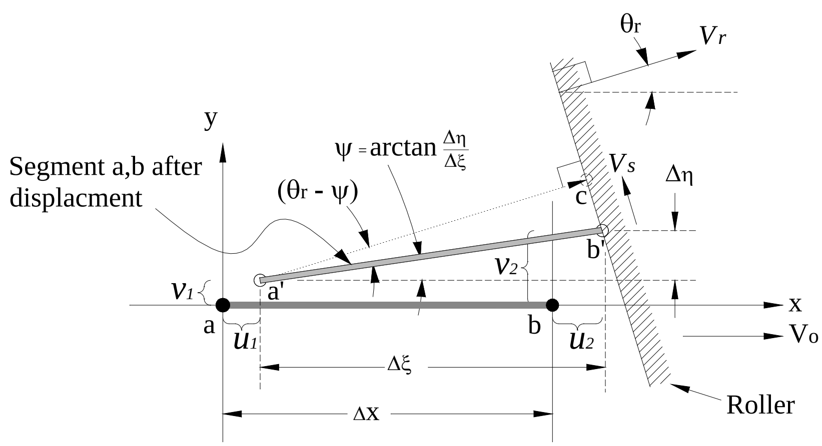

Figure 5
Normal Entry Angle

Referring to Figure 5, a,b is an infinitesimal segment of a particle trajectory in a relaxed web. Segment a',b' is the same segment after the web has started moving, but before it has reached a steady state. The quantities u_1, v_1 and u_2, v_2 may be interpreted as elastic displacements that at any instant of time obey equations (14) and (15). The roller surface velocity controls the speed and direction of web particles very close to the roller. So, all particles on a',b' will move in the direction of V_r. Thus the location of b' will move along the roller face with velocity V_s, equal to,

$$V_s = V_r \tan\left(\theta_r - \psi\right) \qquad (20) \qquad \text{and if } V_s = 0 \qquad \theta_r = \psi \,. \qquad (21)$$

This motion will continue until a steady-state condition is reached where $(\theta_r - \psi) = 0$.

The angle ψ defines the orientation of a vector that is tangent to the particle trajectory. In the deformed coordinates this is,

$$\psi = \tan^{-1}\left(\frac{\partial \eta}{\partial \xi}\right) \approx \frac{\partial \eta}{\partial \xi}\ . \tag{22}$$

Equations (16) may now be used to define the trajectory direction in terms of the *x*, *y* coordinates.

$$\psi = \tan^{-1}\left\{\left[v_x dx + \left(1+\varepsilon_y\right)dy\right]\left[\left(1+\varepsilon_x\right)dx + u_y dy\right]^{-1}\right\} = \theta_r \tag{23}$$

For a uniform web, the relaxed particle trajectories are defined by y = constant. In that case $dy = 0$ and equation (23) becomes,

$$\psi = \tan^{-1} v_x \left(1+\varepsilon_x\right)^{-1} \cong v_x\ . \tag{24}$$

And for uniform web, equation (21) becomes,

$$\tan^{-1} v_x (1+\varepsilon_x)^{-1} \approx v_x = \theta_r\ . \tag{25}$$

A NEW BOUNDARY CONDITION: CONSERVATION OF MASS AND THE NORMAL STRAIN RULE

It will be shown here that application of the principle of conservation of mass, in the context of elasticity theory, leads to a boundary condition that is equal in importance and generality to the normal entry rule.

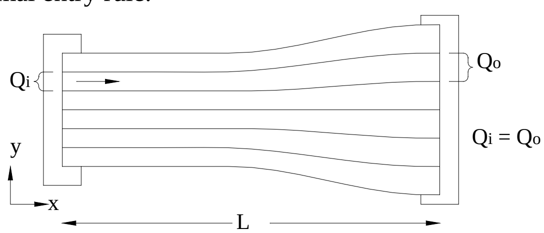

Figure 6
Steady-State Mass Flow

As described earlier, particles traversing a deformed span in a steady state, as illustrated in Figure 6 must follow fixed trajectories. For a uniform web, the trajectories will be straight when it is relaxed. Therefore, if it is marked by a series of such trajectories before deformation, the mass flow rate, Q, between any pair of such curves must be constant. This concept will now be developed using the theory of elasticity.

Let Q_i represent the mass flow rate at the beginning of a path that has a width, dy in the relaxed web. Then,

$$Q_i = \frac{dm}{dt} = V_u dy\left(1+\varepsilon_{yu}\right)h\left(1+\varepsilon_{zu}\right)\rho_u \tag{26}$$

where V_u is the circumferential velocity of the upstream roller, ε_{yu} and ε_{zu} are the strains in the y and z directions at the entry to the upstream roller, h is the web thickness in its relaxed state and ρ_u is the density of the web at the entry to the upstream roller. The density is,

$$\rho_u = \frac{\rho_o}{\left(1+\varepsilon_{xu}\right)\left(1+\varepsilon_{yu}\right)\left(1+\varepsilon_{zu}\right)} \tag{27}$$

where ε_{xu} is the strain at the entry to the upstream roller and ρ_o is the relaxed density of the web. So, Q_i is,

$$Q_i = V_u\left(dy\right)\left(h\right)\left(\rho_o\right)\left(1+\varepsilon_{xu}\right)^{-1} \ . \tag{28}$$

Following the same reasoning, the mass flow at the downstream end of the path is,

$$Q_o = V_d\left(dy\right)\left(h\right)\left(\rho_o\right)\left(1+\varepsilon_{xd}\right)^{-1} \tag{29}$$

where V_d is the circumferential velocity of the downstream roller and ε_{xd} is the strain in the x direction at the downstream roller. Equating Q_o and Q_i

$$\frac{1+\varepsilon_{xd}}{1+\varepsilon_{xu}} = \frac{V_d}{V_u} \tag{30}$$

and solving for ε_{xd},

$$\varepsilon_{xd} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = \frac{V_d}{V_u}\left(1+\varepsilon_{xu}\right) - 1 \ . \tag{31}$$

Thus, the unremarkable fact that the mass of a piece of web doesn't change when it is deformed leads to a boundary condition that seems quite surprising when viewed from the standpoint of elasticity theory:

In a steady state, the ratio of the stretched lengths of an infinitesimal patch of the web at two successive rollers is proportional to the respective ratios of the web velocities at the two rollers (provided the strains and velocities are measured normal to the roller axes). In other words, if the web speeds up by 1% relative to the previous roller, it will have to elongate by 1% to insure that the mass flow is the same at the two locations.

This happens because the cross sectional area and density change by precisely the right amount to satisfy the principle of conservation of mass. In the case of non-uniform rollers, V_u , V_r , ε_{xu} and ε_{xd} are taken to be at the corresponding deformed coordinates ($y + v$) at the respective rollers.

For cases where the roller surface velocity is not aligned with the x-axis, the normal strain rule still applies, provided the strain values are as defined in equations (2) to (4).

A SUMMARY OF THE BOUNDARY CONDITIONS AT A DOWNSTREAM ROLLER

Boundary conditions for steady-state at the entry to a roller may now be summarized as,

$$\tan^{-1}\left[v_x dx + \left(1+\varepsilon_y\right)dy\right]\left[\left(1+\varepsilon_x\right)dx + u_y dy\right]^{-1} = \theta_r \tag{32}$$

$$\varepsilon_x = \frac{V_d}{V_u}\left(1+\varepsilon_o\right) - 1 \ . \tag{33}$$

And for a uniform web, (32) reduces to,

$$\tan^{-1} v_x (1+\varepsilon_x)^{-1} \approx v_x = \theta_r \ . \tag{34}$$

V_d and V_u are, respectively, the downstream and upstream roller surface velocities and it is understood that these may be a function of y. ε_o is the longitudinal strain at the entry to the upstream roller and may also be a function of ($y + v$) if that roller is nonuniform.

It is now apparent that so long as traction is maintained, the controlling conditions at the entry to a roller are fundamentally geometric with stresses only playing a secondary role in controlling the relationships between the strains. This explains why it has been so difficult to find the fourth boundary condition for beam models.

A CONJECTURE ABOUT THE INTERACTION OF THE NORMAL ENTRY AND THE NORMAL STRAIN RULES

The analysis presented here addresses only a steady state condition that meets the boundary conditions (32) and (33). It says nothing, though, about how the web gets there. It is reasonable to ask, therefore, whether there could ever be an initial condition that would cause the web to move away from them – particularly when the web or roller is not uniform. It is conjectured, based on experience that for all of the cases studied in this paper the web always moves from setup conditions toward this state.

PROFILE DISPLACEMENT

In cases where the web is displaced laterally on a nonuniform roller, the lateral translation must be taken into account in determining the effect of the profile. This is called *profile displacement*. The problem may be eliminated in some FEA solvers by using $y + v$ directly in the definition of V_d in equation (33) . When this is not possible, a recursive calculation may be used in which the displacement caused by the profile is used to recalculate the relative profile location.

APPLICATION TO A MISALIGNED DOWNSTREAM ROLLER

Shelton [2] in his 1968 dissertation provided experimental verification of a very effective beam theory model for a misaligned roller. Comparison with his results should, therefore, provide an excellent test of the P.D.E. model.

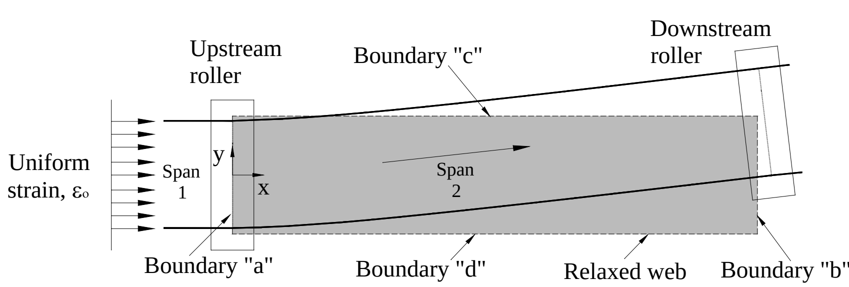

Figure 7
Misaligned downstream roller

<u>Boundary Conditions</u>

Two boundary conditions will be needed for each of the four sides.

Referring to Figure 7, sides “c” and “d” are assumed to be unconstrained. So, for those boundaries the normal and tangential stresses will both be zero. Therefore,

$$\sigma_n = 0 \qquad (35) \qquad\qquad \tau_n = 0 \qquad (36)$$

where σ_n is the stress normal to the boundary and τ_n is the shear stress tangent to the boundary.

Boundary conditions at “b” will be the normal entry and normal strain rules.

$$v_x = \theta_r \qquad (37) \qquad\qquad \varepsilon_x = \frac{V_d}{V_u}(1+\varepsilon_o)-1 \qquad (38)$$

It will be assumed that span 1 is in pure tension with uniform, aligned rollers. So, there will be no boundary defect there. Thus, the boundary at “a” will be a straight line. Furthermore, the displacements at the entry to the upstream roller will determine the boundary conditions at “a”. These are,

$$u = 0 \qquad (39) \qquad\qquad v = -\mu y \varepsilon_o \qquad (40)$$

In theory, *u* should be ε_o times the length of span 1. However, since this will be a constant value, it would produce only a rigid body motion and nothing is lost by setting it to zero.

Comparison with Shelton’s Beam Model

Referencing Shelton’s [2] work, in which he graphed the elastic curves of the web using non-dimensionalized parameters, he divided the problem into two cases depending on the influence of shear stresses - short spans where it is significant and long spans where it can be safely ignored. The parameters are:

$$K_e L \qquad \text{and} \qquad \frac{nT}{AG} \ . \qquad (41)$$

T is the total longitudinal tension, *K* is defined in equation (44) and *n* is a correction factor that accounts for the use of an average value to approximate the shear stress.

$$K_e = \left\{ T \left[EI \left(1 + \frac{nT}{AG} \right) \right]^{-1} \right\}^{0.5} \qquad (42)$$

The static elastic curves for short spans are compared to one set of Shelton’s[2] results in Figure 9 .The solid lines are plots of Shelton’s equations. The data points are from calculations using a partial differential equation solver. The P.D.E. model always includes the influence of shear, even though it is negligible for long spans or low tension. Inputs to the P.D.E. solver are shown in Table 1. For the short span analysis, Shelton mentions that a value of n = 1.2 is often recommended in the literature. A value of 1.0 provided the best agreement at K_eL = 0.1. A value of 1.2 worked better at K_eL = 1. The differences were small, though. For example, at K_eL= 0.1 and *nT/AG* = .005 the difference between 1.0 and 1.2 at *x/L* = 1 was only 1.7%. Theoretically, the ratio of maximum to average shear is 1.5. This was confirmed with the P.D.E model. The fact that lower values provide the best agreement between the two is probably due to other simplifying assumptions in the beam model.

The graph in Figure 8 compares the P.D.E. model with Shelton’s results for long spans. Dimensions and tension were chosen to reduce the effects of shear to negligible levels. For *KL* = 0 the effect of shear is still noticeable as a slight offset.

<table>
<tr><th colspan="10">P.D.E. Model Inputs for Short Spans</th></tr>
<tr><th>W (m)</th><th>$\theta_r{}^o$</th><th>E (Pa)</th><th>μ</th><th>h (mm)</th><th>L (m)</th><th>n</th><th>T (N)</th><th>nT/AG</th><th>KL</th></tr>
<tr><td rowspan="5">1.0</td><td rowspan="5">0.1</td><td rowspan="5">3.1e9</td><td rowspan="5">0.35</td><td rowspan="5">0.1</td><td>15</td><td rowspan="5">1.0</td><td>1.149</td><td>.00001</td><td rowspan="5">0.1</td></tr>
<tr><td>1.062</td><td>229.8</td><td>.002</td></tr>
<tr><td>0.672</td><td>574.6</td><td>.005</td></tr>
<tr><td>0.477</td><td>1149</td><td>.01</td></tr>
<tr><td>0.217</td><td>5746</td><td>.05</td></tr>
<tr><th colspan="10">P.D.E Model Inputs for Long Spans (Selected for Negligible Shear)</th></tr>
<tr><td rowspan="5">1.0</td><td rowspan="5">0.1</td><td rowspan="5">3.1e9</td><td rowspan="5">0.35</td><td rowspan="5">0.1</td><td>15</td><td rowspan="5">1.0</td><td>1.149</td><td>.00001</td><td>0.1</td></tr>
<tr><td>9.534</td><td>1149</td><td rowspan="4">.01</td><td>2</td></tr>
<tr><td>19.07</td><td>1149</td><td>4</td></tr>
<tr><td>28.60</td><td>1149</td><td>6</td></tr>
<tr><td>47.67</td><td>1149</td><td>10</td></tr>
</table>

Table 1
Inputs to Elastic Model for Misaligned Roller

The effect of leaving the nonlinear terms out of equations (14) and (15) would be to cause all of the curves in Figure 8 to look like the one for $K_eL = 0$ or in the case of Figure 9 like the one for $nT/AG = 0$. In other words, without the nonlinear terms the web behaves as though the longitudinal tension has no influence on the elastic curve. It is primarily the addition of $\omega_z\sigma_x$ to the shear in (15) that causes the bending to concentrate nearer the upstream roller.

It is apparent from the agreement in the results that the beam model assumption of zero moment at the downstream roller is confirmed. This can also be seen in the fact that the normal strain rule requires ε_x to be uniform at a uniform downstream roller. The normal entry rule insures σ_y will be small. So, σ_x will be uniform, producing zero moment.

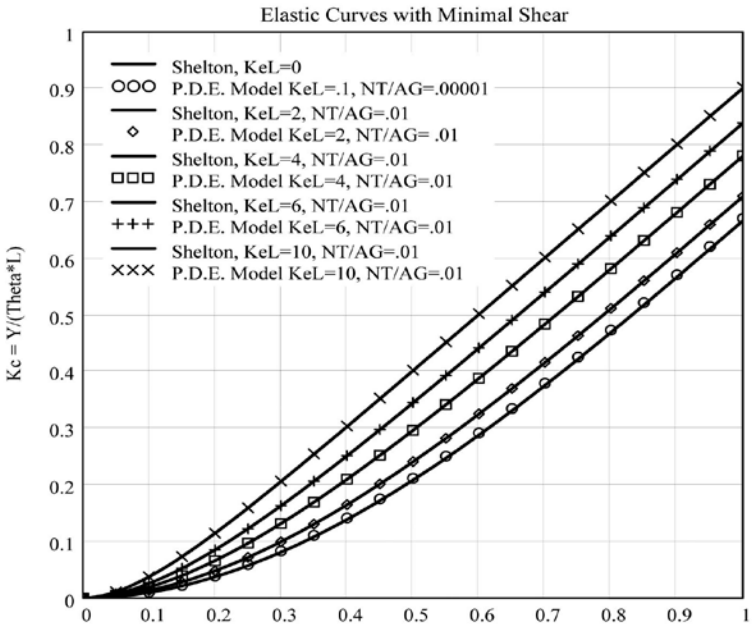

Figure 8
Comparison of P. D. E. Model with Shelton Beam Model for Long Spans

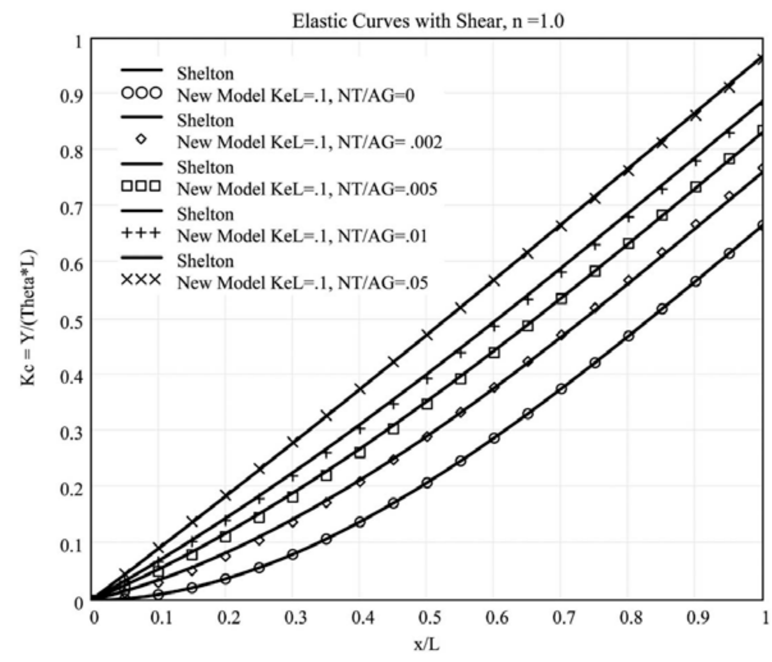

Figure 9
Comparison of P.D.E. Model with Shelton Beam Model for short spans

One of the advantages of a P. D. E. model is that it can produce detailed pictures of stresses throughout the web. An interesting example is the principal minimum stress. The bottom portion of Figure 10 is a contour plot of compressive stress from 0 to –100,000 Pascals (shaded portion). The upper graph shows elevations of the data at 4 positions, spaced at 1 meter intervals down the web starting at the downstream roller. Model inputs were: L = 4 m, W = 1 m, average σ_x = 6.5e6 Pa, θ_r = 0.5 deg, E = 3.1e9 Pa, μ = 0.35. Positive θ_r corresponds to deflection in the $+y$ direction. This information can be used for wrinkle analysis using methods similar to those described by Good, Kedl and Shelton[6].

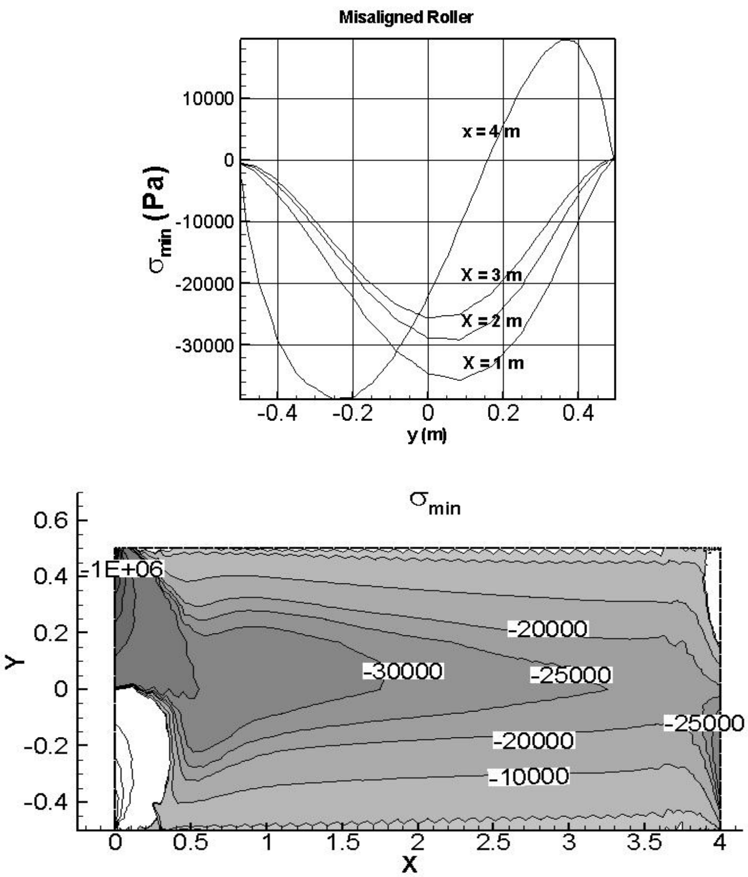

Figure 10
Minimum Principal Stress for a misaligned roller (Pa)

APPLICATION TO A TAPERED ROLLER

Markum and Good [4] performed an experimental evaluation of a concave spreader roller by splitting a web as it exited the upstream roller and measuring the separation at the spreader. Since the experiment is documented well, it provides another good test of the P. D. E. model.

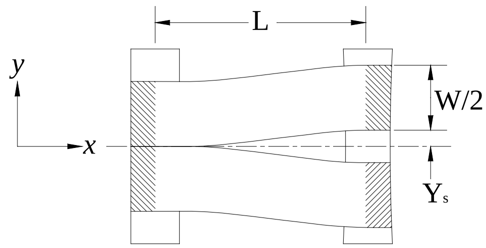

Figure 11
Markum and Good test

Boundary Conditions

For the downstream boundary the normal strain and normal entry rules are.

$$v_x = \theta_r \quad (43)$$

$$\varepsilon_x = \frac{V_d}{V_u}\left(1+\varepsilon_o\right)-1 \quad (44)$$

Equation (44) may be put into a more convenient form if V_d is expressed as a fraction of V_u. And since a non-uniform roller will cause this ratio to vary with distance from the roller centerlines, V_d can be expressed as a function of the fractional difference in roller diameters. If *f(y)* is defined as,

$$f(y) = \frac{D_d\left(y\right)-D_u}{D_u} \quad (45)$$

where, D_d and D_u are the respective diameters of the downstream and upstream rollers and y is the distance from the roller centerlines. Then,

$$V_d = V_u\ \left[1+f\left(y\right)\right] \quad (46) \qquad \text{and} \qquad \varepsilon_x = \left(1+\varepsilon_o\right)\left(1+f(y)\right)-1 \cong \varepsilon_o + f\left(y\right) \quad .$$
(47)

Note that the deformed *y* coordinate $\eta = y + v$ may be used as the independent variable for *f(y)* in cases where more accuracy is desired and the solution method allows it. This is done for the results reported here.

The conditions at boundary "d" will, as in the previous cases, be:

$$v = -\mu\varepsilon_o y \quad (48) \qquad\qquad u = 0 \quad (49)$$

The concave roller surface in the Markum and Good experiment was cut with a circular arc of radius R_o = 10.16m. The diameter is approximated mathematically as,

$$D_d\left(y\right) = 2\left(\frac{D_c}{2}+\frac{y^2}{2R_o}\right) = 2\left(0.02776+0.0492y^2\right) \quad (50)$$

Their profile was expressed in terms of roller radius and $D_c/2$ is the midpoint radius. If the upstream roller turns freely, the average of V_d/V_u is zero and *f(y)* will be,

$$f\left(y\right) = 1.773m^{-2}y^2 - 0.003414 \quad (51)$$

Comparison of Results

The test arrangement is shown in Figure 11. Roller profiles were either circular arcs or "bow tie" and the test materials were either PET or LDPE. The bow tie rollers had uniform diameter in the middle with tapered edges. Problems with traction and yielding were reported for many of the tests. The best results, shown in their Figure 5, was the "parabolic 2" profile with LDPE. It is examined in detail. Relative to a web edge at 0.076m from the midpoint of the roller, "Parabolic 2" had a profile depth of 0.28 mm.

For each test, three sets of results are presented. One is data from the Markum and Good beam model. The second is from an enhancement of the Markum and Good calculation using a recursive method to compensate for profile displacement in which the calculated value of Y_s is added to the y coordinate and then Y_s is recalculated. The solution converged to a steady value in six repetitions. The third is the from the P. D. E. model. The measured values were scaled off the graph in Figure 5 of their paper.

Test parameters are described in Table 2. Results are in Table 3 .

#	Material	Modulus (MPa)	Caliper (μm)	Width (m)	Length (m)	Tension (N)	Profile Radius (m)	Profile f(y)
1	LDPE	165.5	25.4	0.152	0.419	8.9	10.16	$.02776 + .0492y^2$
2	"	"	"	"	0.419	17.8	"	"
3	"	"	"	"	0.242	8.9	"	"
4	"	"	"	"	0.242	17.8	"	"

Table 2
Test parameters for Markum and Good experiment

#	Measured Separation 2(Ys) (m)	Beam Model (m)	Pct Error relative to measured	Recursive beam model (m)	Pct Error relative to measured	P. D. E. model (m)	Pct Error relative to measured
1	.0079	0.0072	-8.3%	0.00800	1.33%	.00792	0.253%
2	.0075	.0067	10.4	0.00737	-1.74	.00702	-6.40
3	.0031	.0026	-17.8	0.00264	-14.9	.00274	-11.6
4	.0027	.0024	-10	0.00256	-5.03	.00255	-5.56

Table 3
Comparison of results with the three models

The improved accuracies of the recursive beam and the P. D. E. models are due to compensation for profile displacement.

The reasons for success of the beam model are apparent in detailed plots from the P. D. E. model shown in Figure 12 and Figure 13. The assumption that $\sigma_y << \sigma_x$, used to estimate the end moment, is valid. And though there are large areas of compressive stress (shown shaded), the levels are generally low except in a small area near the upstream roller. The minimum principal stress, σ_{min} , plotted in Figure 13 is very nearly equal to σ_y .

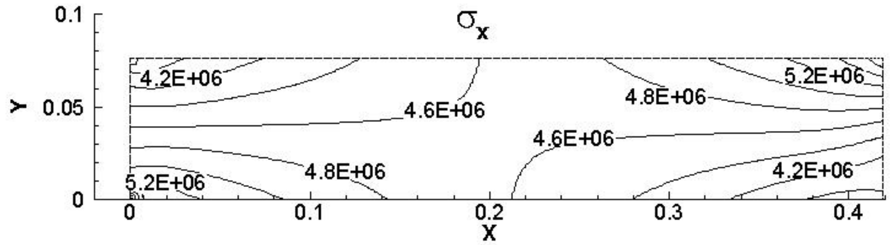

Figure 12
Longitudinal stress for test # 2 (Pa)

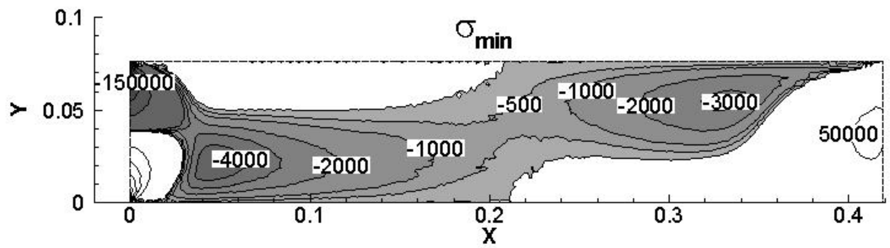

Figure 13
Minimum principal stress for test # 2 (Pa)
Areas of negative stress shown shaded

CONCLUSIONS

A new method for solving problems involving deformation and translation of moving webs is now available.

- It is founded on basic elasticity theory in a way that permits the use of general-purpose numerical methods to rapidly solve the partial differential equations.
- It introduces a rigorous definition of the normal entry rule suitable for use with elasticity theory.
- It introduces a new boundary condition for the downstream roller that has the same range of application as the normal entry rule.
- It shows that so long as traction is maintained, the controlling conditions at the entry to a roller are fundamentally geometric with stresses only controlling the relationships between the strains that govern particle paths and mass flow.
- It has been shown to produce solutions that are in agreement with experimental results reported for both misaligned and tapered rollers.
- It is capable of providing detailed descriptions of stress and deformation fields throughout web spans.
- It can be used to identify relationships that help in the creation of simplified models.

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