A Comparison of Multi-Span Lateral Dynamics Models

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- In her 1987 thesis and a follow up paper in the IEEE Transactions, Lisa Sievers described a multi-span model based on the Timoshenko beam and showed that it performed better than a similar model based on the Euler Bernoulli beam.
- Then, in 1989 Young, Shelton & Kardamilas (YSK) [] published a description of a new Euler-Bernoulli multi-span model. It transfer's lateral bending deformation across rollers and is functionally equivalent to the Sievers Euler Bernoulli model. A notable feature of the YSK model is the way it uses transfer functions to interconnect the spans.

Introduction

- The original goal for this paper was to recast the Sievers Timoshenko model into the same analytical form as the YSK model, develop a similar interconnection strategy and then compare the Euler Bernoulli and Timoshenko versions quantitatively.
- A YSK-type Timoshenko model has been developed, and it looks quite plausible. However, it produces a value for the curvature factor that doesn't make sense. After exhaustive troubleshooting, I've concluded that the problem is not due to a procedural error; but is more likely something of a conceptual nature.

Introduction (Cont.)

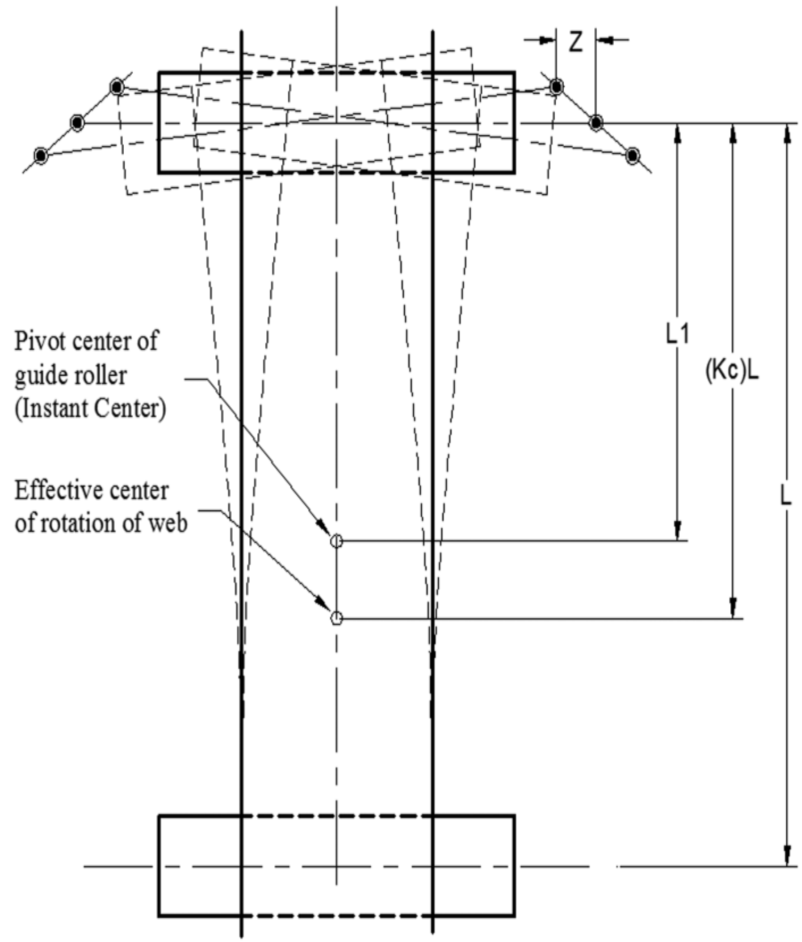
- As an alternative to the original plan I will
  - Describe the problem with the YSK-type Timoshenko model
  - Answer the question, “How much difference does shear make.” by making a quantitative comparison between the Euler-Bernoulli model of the YSK paper and a modified version of the Sievers Timoshenko model.
- A detailed derivation of the modified Sievers Timoshenko model is described in a companion paper presented at this conference.

Notation

- Sievers referenced all of her variables to rollers using superscripts to indicate whether a variable was defined at the upstream or downstream side of a particular roller.
- In this paper and much of the current literature, variables are referenced to spans. A variable labeled y_{02} would indicate the value of y at $x = 0$, in span 2. A variable labeled y_{L2} would indicate the value of y at $x = L$, in span 2.

Curvature factor

The curvature factor is a good indicator of model validity because it relies in an intimate way on the shape of the web when it is controlled by the normal entry rule. Shelton derived and published it for his simplest single span model in 1968 and it shows up in more sophisticated dynamic models as the static gain factor in transfer functions.

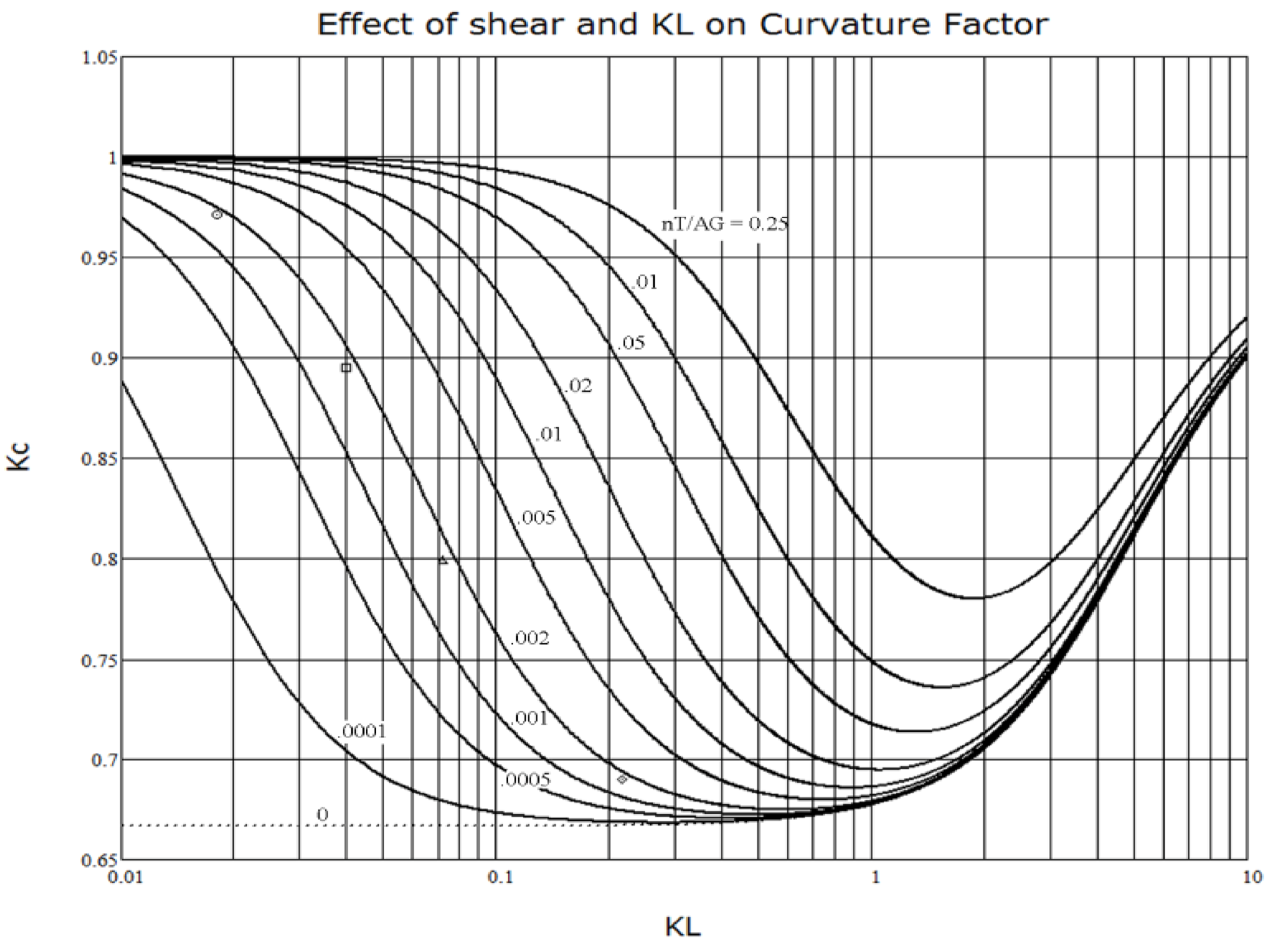


Curvature factor (Cont.)

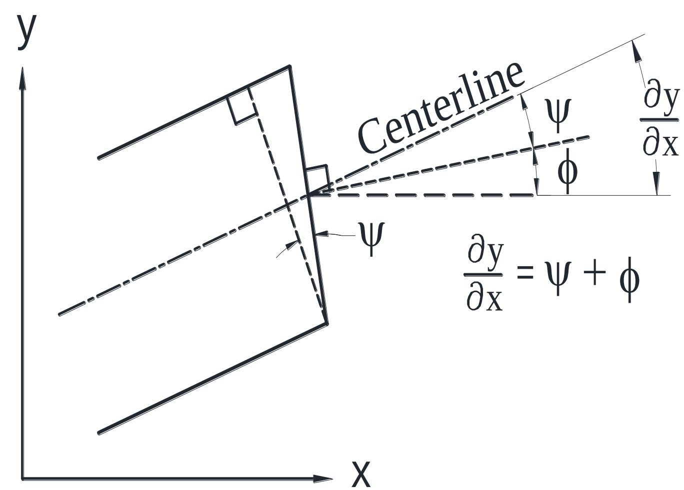
$$K_c = \frac{1}{KL}\frac{\sinh(KL) - KLa\cosh(KL)}{1 - a\cosh(KL)} \qquad K = \sqrt{\frac{T}{EIa}} \qquad a = 1 + \frac{nT}{AG}$$

T is tension in units of force, E is the elastic modulus, I is the area moment of inertia and G is the shear modulus. For the Timoshenko beam $n = 1.2$. For a Euler Bernoulli beam the same equations are used, but $n = 0$ and, consequently, $a = 1$.

Curvature factor (Cont.)



The effect of shear



The static equations for web shape

$$y' = \phi + \psi$$

$$\left(\frac{AG}{n} + T\right) y'' - \frac{AG}{n}\phi' = 0$$

$$EI\phi'' + \frac{AG}{n}\left(y' - \phi\right) = 0$$

From these a familiar equation is derived.

$$\frac{d^4 y}{dx^4} - K^2 \frac{d^2 y}{dx^2} = 0 \qquad K^2 = \frac{T}{EI\left(1 + \frac{nT}{AG}\right)}$$

Whose solution is,

$$y(x) = C_1 \sinh\left(Kx\right) + C_2 \cosh\left(Kx\right) + C_3 x + C_4$$

Shape equation

$$y\big|_{x=0} = y_0 \qquad y\big|_{x=L} = y_L$$

$$\phi\big|_{x=0} = \phi_0 \qquad \phi\big|_{x=L} = \phi_L$$

Boundary conditions for Timoshenko

Solve for the coefficients C_1, C_2, C_3 & C_4. Then, rearrange solution in the form of products of boundary conditions and shape factors.

$$y(x) = y_0 + \left(y_0 - y_L\right) g_4(x,L) + \phi_L g_5(x,L) + \phi_0 g_6(x,L)$$

Shape factors

$$g_4(x) = \frac{\cosh(Kx)+\cosh(KL)-\cosh(KL-Kx)-Kax\sinh(KL)-1}{KLa\sinh(KL)-2\left(\cosh(KL-1\right)}$$

$$g_5(x) = \frac{KLa\left(\cosh(Kx)-1\right)-Kax\left(\cosh(KL)-1\right)-\sinh(Kx)-\sinh(KL-Kx)+\sinh(KL)}{Ka\left[KLa\sinh(KL)-2\left(\cosh(KL-1\right)\right]}$$

$$g_6(x) = \frac{\sinh(Kx)-\sinh(KL)+\sinh(KL-Kx)-KLa\left(\cosh(KL-Kx)-1\right)+Ka(L-x)(\cosh(KL)-1)}{Ka\left[KLa\sinh(KL)-2\left(\cosh(KL-1\right)\right]}$$

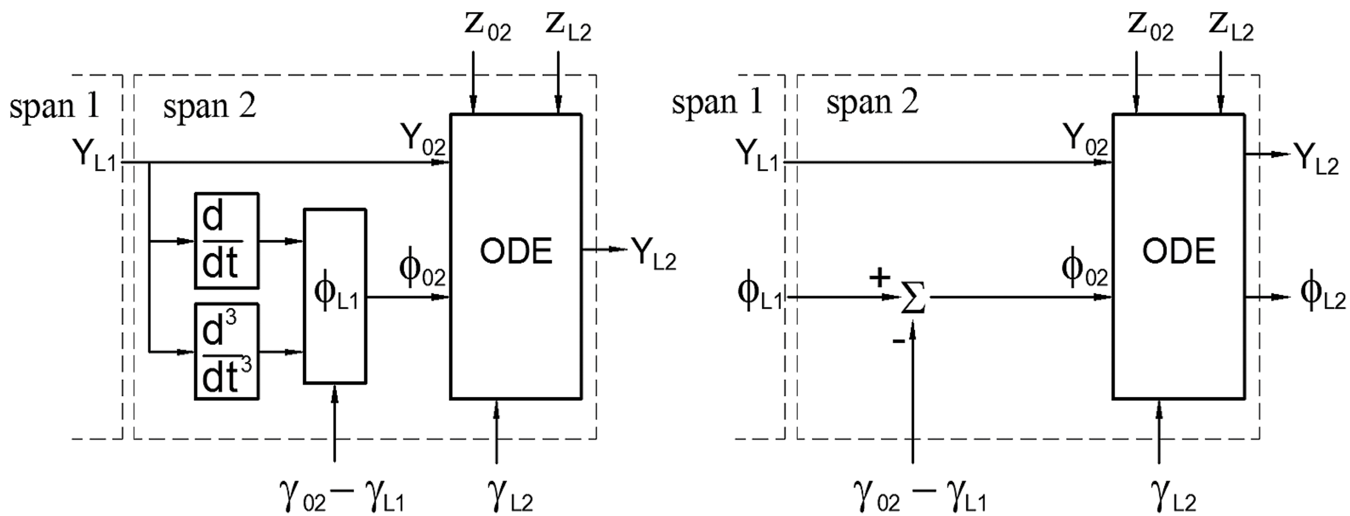
Fork in the road

The Sievers model didn't concern itself with the details of ϕ. It was computed as a single number in the previous span and passed on to the next, but to get to transfer functions, it will be necessary to know its value in terms of y. From the governing equations it can be shown that,

$$\phi = \frac{dy}{dx} + L^2 b \frac{d^3 y}{dx^3}$$

where $$b = \frac{1}{6}\frac{W^2}{L^2} n \left(1+\mu\right) a$$

Here’s how the YSK method works



YSK method   Sievers method

In the YSK method, only the lateral displacement is passed to the next span. Since all of the information about its derivatives is implicit in the lateral displacement, it is possible to reconstruct the cross section rotation from the previous span by mathematical manipulation.

Get an expression for curvature

Second derivative of shape equation

$$\left.\frac{d^2}{dx^2}y(x)\right|_L = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \phi_L \frac{g_2}{L} + \phi_0 \frac{g_3}{L}$$

$$g_1 = L^2\left(g_4''(L)\right) = \frac{K^2L^2a\left(\cosh(KL)-1\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$g_2 = L\left(g_5''(L)\right) = \frac{KL\left(KLa\cosh(KL)-\sinh(KL)\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$g_3 = L\left(g_6''(L)\right) = \frac{KL\left(\sinh(KL)-KLa\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

Substitute expression for ϕ

$$\phi = \frac{dy}{dx} + L^2 b \frac{d^3 y}{dx^3}$$

$$\left.\frac{d^2}{dx^2} y(x)\right|_L = \left(y_0 - y_L\right) \frac{g_1}{L^2} + \left(\frac{dy_L}{dx} + L^2 b \frac{d^3 y_L}{dx^3}\right) \frac{g_2}{L} + \left(\frac{dy_0}{dx} + L^2 b \frac{d^3 y_0}{dx^3}\right) \frac{g_3}{L}$$

Now all that is needed is to convert spatial derivatives to time derivatives.

Relations between spatial and time derivatives

The normal entry rule provides first order relation

$$\left.\frac{\partial y_L}{\partial x}\right|_{x=L} = \frac{1}{v_o}\left(\frac{dz_L}{dt} - \frac{dy_L}{dt}\right) + \gamma_L$$

Using the chain rule, the acceleration is

$$\frac{d^2 y}{dt^2} = \frac{\partial^2 y(x,t)}{\partial t^2} + 2\frac{\partial y(x,t)}{\partial x \partial t} v_o + \frac{\partial^2 y(x,t)}{\partial x^2} v_o^{\ 2} = 0$$

The cross derivative can be eliminated by taking the spatial derivative of the normal entry rule.

$$\frac{\partial^2 y_L}{\partial x \partial t} = -v_o \left.\frac{\partial^2 y_L}{\partial x^2}\right|_{x=L}$$

Relations between derivatives of position and time (Cont.)

$$\frac{d^2 y_L}{dt^2} = v_o{}^2 \left(\left. \frac{\partial y_L^2}{\partial x^2} \right|_{x=L} \right)$$

Note: *In the steady state, the time derivative on the left goes to zero and this becomes a statement of the steady state 4th boundary condition discovered by Shelton*.

A similar procedure for the third derivative yields,

$$\frac{d^3 y_L}{dt^3} = -v_o{}^3 \left(\left. \frac{\partial^3 y_L}{\partial x^3} \right|_{x=L} \right)$$

Summary of position - time relationships

These are used to animate the shape equation by making the boundary conditions a function of time.

$$\frac{\partial y}{\partial x} = v_o\left(\gamma - \frac{\partial y}{\partial t}\right)$$

Velocity (Normal entry rule)

$$\frac{\partial^2 y_L}{\partial x^2} = \frac{1}{{v_o}^2}\left(\frac{\partial^2 y_L}{\partial t^2} - \frac{\partial^2 z_L}{\partial t^2}\right)$$

Acceleration

$$\frac{\partial^3 y_L}{\partial x^3} = \frac{1}{{v_o}^3}\left(\frac{\partial^3 z_L}{\partial t^2} - \frac{\partial^3 y_L}{\partial t^2}\right)$$

Jerk

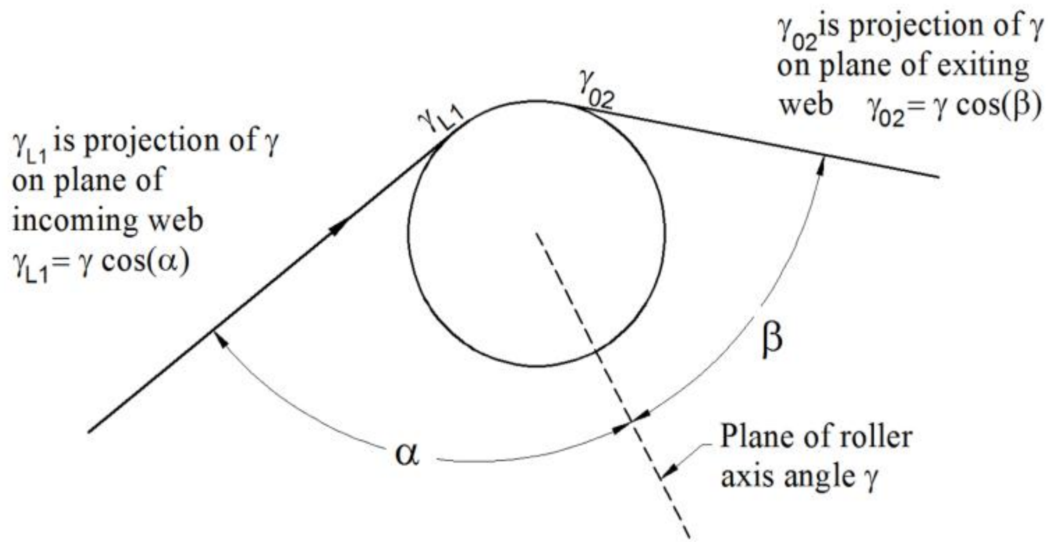
Time-based ODE

$$\frac{1}{v_o^2}\left(\frac{d^2 y_L}{dt^2}-\frac{d^2 z_L}{dt^2}\right)=\left(y_0-y_L\right)\frac{g_1}{L^2}+\left[\frac{1}{v_o}\left(\frac{dz_L}{dt}-\frac{dy_L}{dt}\right)+\gamma_L+L^2 b\frac{1}{v_o^3}\left(\frac{\partial^3 z_L}{\partial t^3}-\frac{\partial^3 y_L}{\partial t^3}\right)\right]\frac{g_2}{L}$$

$$+\left[\frac{1}{v_o}\left(\frac{dz_0}{dt}-\frac{dy_0}{dt}\right)+\gamma_0+L^2 b\frac{1}{v_o^3}\left(\frac{\partial^3 z_0}{\partial t^3}-\frac{\partial^3 y_0}{\partial t^3}\right)\right]\frac{g_3}{L}$$

There is one last issue to resolve. If the upstream roller is misaligned (either accidentally or because it is part of a guiding system), γ_o will be nonzero and will not be the same as γ_L in the previous span.

Effect of roller axis angle on γ_0



$$\gamma_{L1} = \gamma \cos(\alpha) \quad \text{and} \quad \gamma_0 = \gamma \cos(\beta)$$

The transfer function

$$y_L = y_0 \frac{\left(-\tau b g_3 s^3 - \frac{g_3}{\tau} s + \frac{g_1}{\tau^2}\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)} + z_L \frac{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)}$$

$$+ z_0 \frac{\left(\frac{g_3}{\tau} s + \tau b g_3 s^3\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)} + \gamma_L \frac{\left(\frac{v_o}{\tau} g_2\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)}$$

$$+ \gamma_0 \frac{\left(g_3 \frac{v_o}{\tau}\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)}$$

Three tests

There are three tests that can be applied to the preceding transfer function.

Test 1: Defaults to Euler Bernoulli when n = 0 - pass

Test 2: Displacement guide with z input has no dynamics - pass

Test 3: Curvature factor consistent with static gain – fail

The curvature factor issue

When γ_L in the transfer function is equal to $z_L/(K_c L)$, and all other inputs are zero, the static gain for the transfer function should be unity and K_c will be

$$\tilde{K}_c = \frac{g_2}{g_1} \qquad \tilde{K}_c = \frac{\left(KLa\cosh(KL) - \sinh(KL)\right)}{KLa\left(\cosh(KL) - 1\right)}$$

K_c ought to be:

$$K_c = \frac{KLa\cosh\left(KL\right) - \sinh\left(KL\right)}{KL\left(a\cosh\left(KL\right) - 1\right)}$$

The position of a in the denominator makes a big difference.

The curvature factor issue (Cont.)

The value of $\tilde{K}_c$ disagrees with two other sources.

- The static gain calculated for the modified Sievers Timoshenko beam.
- The value calculated for a static Timoshenko beam by Shelton .

It is interesting to note that it reduces to the correct value for the Euler Bernoulli beam, when $a = 1$.

L =	22 inches
T =	44.5 lbf
h =	0.0034 inches
E =	550,000 psi
W =	44.5 inches

$$K_c = 0.89$$

$$\tilde{K}_c = 2.5$$

Shelton's dynamic Timoshenko model

It used an acceleration equation with an extra "shear" term.

$$\frac{d^2 y_L}{dt^2} = v_o{}^2 \left(\left. \frac{\partial^2 y}{\partial x^2} \right|_{x=L} \right) + \frac{d^2 z}{dt^2} - v_o \frac{d\theta_{Ls}}{dt}$$

It also produced an unrealistic curvature factor. $\hat{K}_c = 5.2$

He must have had concerns about the validity of this model because he didn't produce plots of its frequency response as he did for everything else.

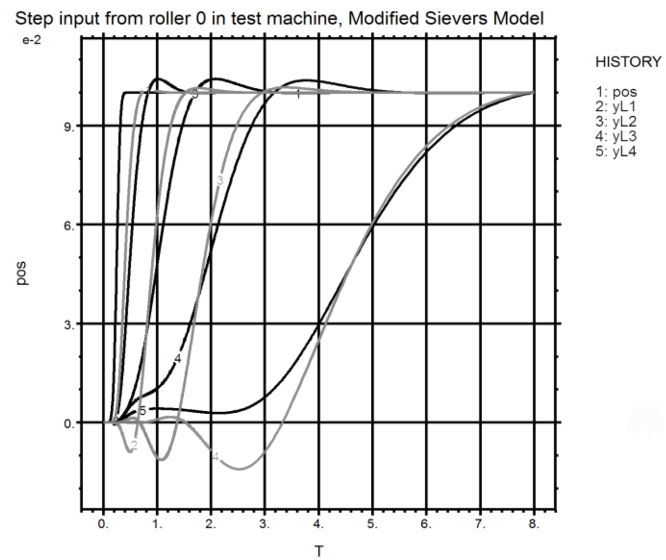
So, how much difference does shear make?

The only dynamic lateral model with shear that I trust is a modification of one based on Sievers' work. It passed all three of the tests described above. Sievers' original model and the modified model are described in the companion to this paper.

An example of the difference between the Timoshenko and Euler Bernoulli models is shown on the next page. Parameters are:

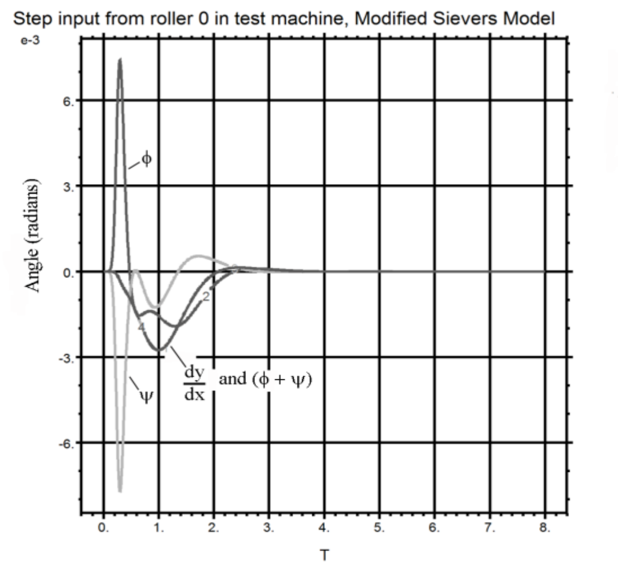
v_o = 200 fpm, T = 44.5 lbf, h = 0.0034 inches
E = 550,000 psi, W = 44.5 inches, μ = 0.3

A simulated example



The four spans have lengths of 10, 22, 40 and 120 inches. Curves show the lateral displacements at the downstream ends of each one. The dark lines are for the Timoshenko model and the gray lines are for the Euler Bernoulli model. The leftmost curve is the input displacement at the upstream roller of the first span.

A simulated example (Cont.)



This graph shows ϕ, ψ and dy_L/dx for the downstream end of the second span. The sum of ϕ and ψ is also plotted, but isn't visible because it is identical to the curve for dy_L/dx (as it should be). It is clear that for this span, ϕ, ψ and dy_L/dx are comparable to one another in magnitude.

TRACTION

All of the simulations and analysis described here assume that the web becomes locked to a roller surface at the line of entry and stays locked until it reaches the exit, but we all know that this isn't true – especially during transient conditions. It was for this reason that I was frankly surprised that Sievers' model worked as well as it did. This is an area of web handling that needs more attention.

CONCLUSIONS

- A YSK-type Timoshenko model has been developed, and it looks quite plausible. However, it produces a value for the curvature factor that doesn't make sense. After exhaustive troubleshooting, I've concluded that the problem is most likely something of a conceptual nature.

Conclusions (Cont.)

- The modified Sievers' model was used to compare multi-span models with and without shear. The simulations indicate that there are nontrivial differences.
  - In short spans ϕ, ψ and dy_L/dx can be comparable to one another in magnitude.
  - For short spans there are significant qualitative differences in response to a step input.
  - The optimum time response for a remote pivot steering guide occurs when the radius of rotation of the guide mechanism equals K_cL. In short spans, shear significantly affects K_c.

Conclusions (Cont.)

- The simulated response to a step input is suggestive of a straightforward experiment which could shed considerable light on lateral dynamics models. It would be a simple matter to arrange a displacement guide to produce a step input into a series of spans instrumented to monitor their displacements.