A Belated Appreciation of Lisa Sievers' Thesis

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Why review it?

- My impression is that it was not especially well-regarded.
- But when I read it for the first time last year, I found much to admire.
  - Mathematical rigor
  - A workable lateral dynamic model that includes shear

Three features caught my attention

- Use of Hamilton’s principle for deriving the governing equations.
- Derivation of the normal entry rule by application of the material derivative.
- Use of spectral separation to justify use of static web shape in a dynamic model.

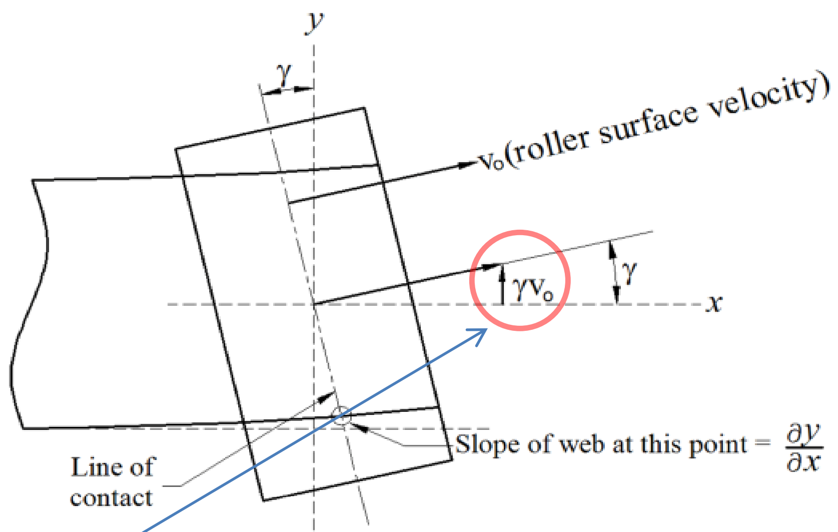
Three dynamic multi-span models

- Convecting string with zero bending stiffness
- Euler-Bernoulli beam with bending stiffness and no shear
- Timoshenko beam with both bending and shear

It produced a bonus

- I believe Sievers' Timoshenko model is sound.
- The model she implemented could not handle single misaligned rollers, however, all the pieces were there to implement a more general model.

Normal entry rule



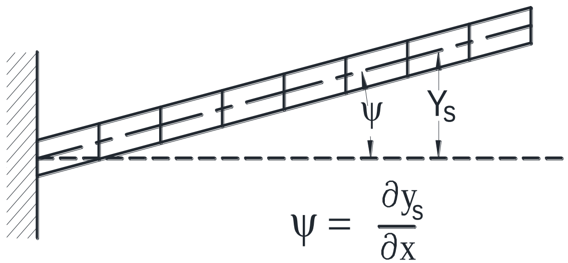
Lagrangian velocity

Eulerian velocity

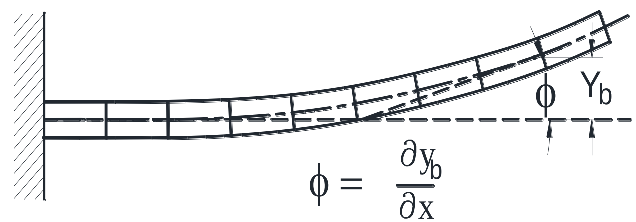
$$\frac{dy}{dt} = \gamma v_o = \frac{\partial y}{\partial t} + v_o \frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial t} = v_o \left(\gamma - \frac{\partial y}{\partial x} \right)$$ Normal entry rule

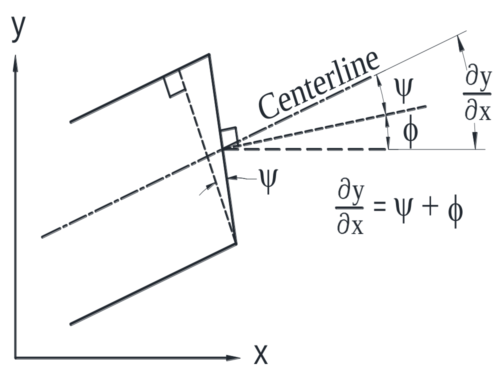
Relationship of shear & bending



Pure shear deflection



Pure bending deflection



Φ is the "cross section rotation".
Sievers called it "face angle".

Hamilton's principle

- Hamilton's Principle uses calculus of variations to find a pair of coupled equations that minimize K-V

$$K = \frac{1}{2}\int_0^L m\left(v_o{}^2 + \left(\frac{\partial y}{\partial t} + v_o\frac{\partial y}{\partial x}\right)^2\right)dx + \frac{1}{2}\int_0^L J\left(\frac{\partial \phi}{\partial t} + v_o\frac{\partial \phi}{\partial x}\right)^2 dx$$

$$V = \frac{1}{2}\int_0^L EI\left(\frac{\partial \phi}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L \frac{AG}{n}(\psi)^2 dx + \frac{1}{2}\int_0^L T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Governing equations (Cont.)

$$y' = \phi + \psi$$

$$-m\left(\ddot{y}+2v_O\dot{y}'+v_O{}^2 y''\right)+\left(\frac{AG}{n}+T\right)y''-\frac{AG}{n}\phi'=0$$

$$J\left(\ddot{\phi}+2v_O\dot{\phi}'+v_O{}^2\phi''\right)+EI\phi''+\frac{AG}{n}\left(y'-\phi\right)=0$$

Eulerian accelerations.
Spectral separation is used to
justify ignoring these terms.

Spectral separation

- Means that natural frequencies of elastic system are significantly higher than its operating frequencies.
- Sievers used the frequency of the lowest mode of the string model as a lower bound. She stated that it was 70 Hz. I calculated 13 Hz.

Spectral separation (Cont.)

- To get a little more insight, I calculated the fundamental mode frequency of a stationary, tensioned Euler Bernoulli beam using parameters for the last span on the test machine. Assuming clamped ends, it is *46 Hz*.
- Similar calculation for Timoshenko beam gave 28 Hz.
- These frequencies are lower than expected, but her test frequencies were all less than 0.07 Hz. So, her criterion of an order of magnitude separation was met.

Three static equations

$$y' = \phi + \psi$$

$$\left(\frac{AG}{n} + T\right) y'' - \frac{AG}{n} \phi' = 0$$

$$EI\phi'' + \frac{AG}{n}\left(y' - \phi\right) = 0$$

From these a familiar equation is derived.

$$\frac{d^4 y}{dx^4} - K^2 \frac{d^2 y}{dx^2} = 0 \qquad K^2 = \frac{T}{EI\left(1 + \frac{nT}{AG}\right)}$$

Whose solution is,

$$y(x) = C_1 \sinh\left(Kx\right) + C_2 \cosh\left(Kx\right) + C_3 x + C_4$$

Shear & cross section rotation

$$\psi = -EIa \frac{n}{AG} \frac{d^3 y}{dx^3}$$

$$\phi = \frac{dy}{dx} + EIa \frac{n}{AG} \frac{d^3 y}{dx^3}$$

$$a = 1 + \frac{nT}{AG}$$

The shape equation

$$y\big|_{x=0} = y_0 \qquad y\big|_{x=L} = y_L$$

$$\phi\big|_{x=0} = \phi_0 \qquad \phi\big|_{x=L} = \phi_L$$

Boundary conditions for Timoshenko

Solve for the coefficients C_1, C_2, C_3 & C_4. Then, rearrange solution in the form of products of boundary conditions and shape factors.

$$y(x) = y_0 + \left(y_0 - y_L\right) g_4(x,L) + \phi_L g_5(x,L) + \phi_0 g_6(x,L)$$

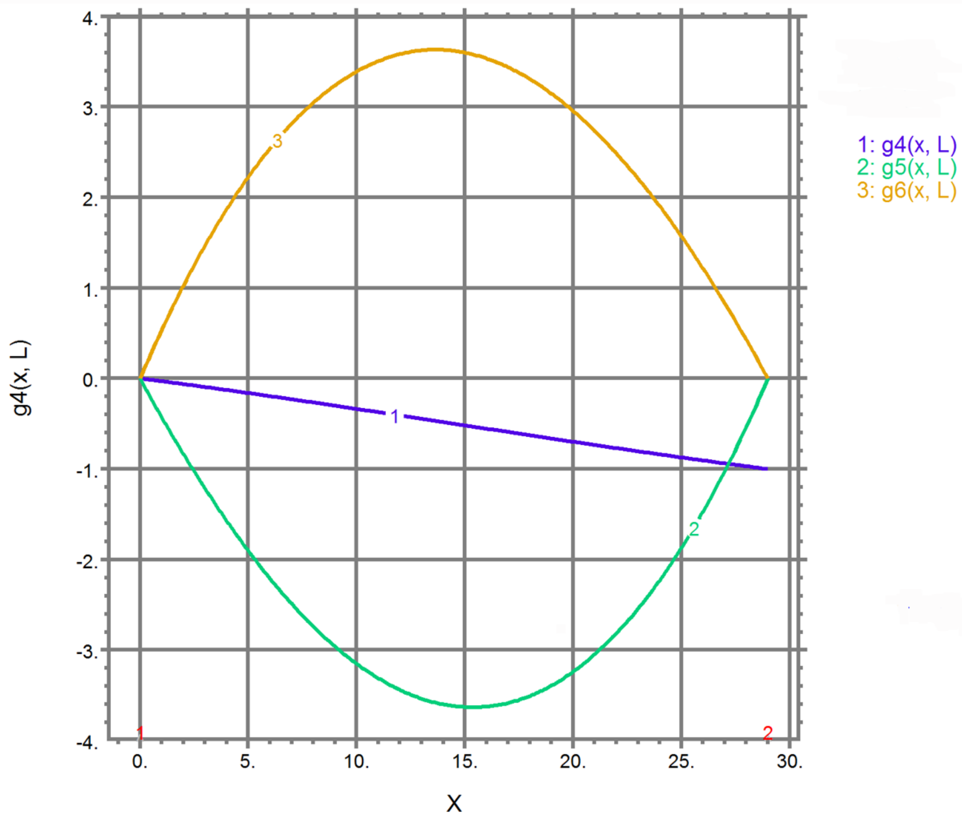
Shape factors

$$g_4(x) = \frac{\cosh(Kx)+\cosh(KL)-\cosh(KL-Kx)-Kax\sinh(KL)-1}{KLa\sinh(KL)-2\left(\cosh(KL-1\right)}$$

$$g_5(x) = \frac{KLa\left(\cosh(Kx)-1\right)-Kax\left(\cosh(KL)-1\right)-\sinh(Kx)-\sinh(KL-Kx)+\sinh(KL)}{Ka\left[KLa\sinh(KL)-2\left(\cosh(KL-1\right)\right]}$$

$$g_6(x) = \frac{\sinh(Kx)-\sinh(KL)+\sinh(KL-Kx)-KLa\left(\cosh(KL-Kx)-1\right)+Ka(L-x)(\cosh(KL)-1)}{Ka\left[KLa\sinh(KL)-2\left(\cosh(KL-1\right)\right]}$$

Shape factors (Cont.)



Relations between spatial and time derivatives

These are used to animate the shape equation by making the boundary conditions a function of time.

Movement relative to roller

Movement of roller

$$\frac{\partial y}{\partial t} = v_o\left(\gamma - \frac{\partial y}{\partial x}\right) + \frac{dz}{dt}$$

Velocity (Normal entry rule)

$$\frac{\partial^2 y}{\partial t^2} = v_o{}^2 \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 z}{\partial t^2}$$

Acceleration

Plan is to create a time-based ODE that is a function of y, derivatives of y and ϕ_L from the previous span.

First step:
Get an expression for curvature

Second derivative of shape equation

$$\left.\frac{d^2}{dx^2}y(x)\right|_L = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \phi_L\frac{g_2}{L} + \phi_0\frac{g_3}{L}$$

$$g_1 = L^2\left(g_4''(L)\right) = \frac{K^2L^2a\left(\cosh(KL)-1\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$g_2 = L\left(g_5''(L)\right) = \frac{KL\left(KLa\cosh(KL)-\sinh(KL)\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$g_3 = L\left(g_6''(L)\right) = \frac{KL\left(\sinh(KL)-KLa\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

Second step:
Get an expression for ϕ_L

First derivative of shape equation

$$\frac{d}{dx} y(x)\bigg|_L = \left(y_0 - y_L\right)\frac{h_1}{L} + \phi_L h_2 + \phi_0 h_3$$

Solve for ϕ_L

$$\phi_L = \frac{1}{h_2}\left(\frac{\partial y_L}{\partial x} - h_3 \phi_0 - \frac{h_1}{L}\left(y_0 - y_L\right)\right)$$

$$h_1 = L\left(g_4{}'(L)\right) = \frac{KLa\sinh(KL)(1-a)}{a\left[KLa\sinh(KL) - 2\left(\cosh(KL) - 1\right)\right]}$$

$$h_2 = g_5{}'(L) = \frac{(a+1)(1-\cosh(KL)) + KLa\sinh(KL)}{a\left[KLA\sinh(KL) - 2\left(\cosh(KL) - 1\right)\right]}$$

$$h_3 = g_6{}'(L) = \frac{(a-1)(1-\cosh(KL))}{a\left[KLA\sinh(KL) - 2\left(\cosh(KL) - 1\right)\right]}$$

Third step: Substitute ϕ_L into curvature equation

$$\frac{d^2 y_L}{dx^2} = \left(y_0 - y_L\right)\frac{1}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\frac{1}{L}\left[\frac{dy_L}{dx}\right] + \frac{\phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right)$$

ϕ_0 is equal to ϕ_L from previous span.

Now all that is needed is to convert spatial derivatives to time derivatives.

Fourth step: Replace spatial derivatives with time derivatives

Use velocity and acceleration relationships to replace the spatial derivatives.

$$\frac{\partial y_L}{\partial x} = \frac{1}{v_o}\left(\frac{dz_L}{dt} - \frac{\partial y_L}{\partial t}\right) + \gamma_L \qquad \frac{d^2 y_L}{dx^2} = \frac{1}{{v_o}^2}\left(\frac{d^2 y_L}{dt^2} - \frac{d^2 z_L}{dt^2}\right)$$

$$\frac{d^2 y_L}{dt^2} = \left(y_0 - y_L\right)\frac{{v_o}^2}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\left[\frac{v_o}{L}\left(\frac{dz_L}{dt} - \frac{dy_L}{dt}\right) + \frac{{v_o}^2}{L}\gamma_L\right]$$
$$+ \frac{{v_o}^2 \phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right) + \frac{d^2 z_L}{dt^2}$$

Sievers had this equation in the thesis, but without γ_L , and used it only for fixed parallel rollers.

Solving the equation

In the previous equation, when the values of y_o and ϕ_o depend on y_L and ϕ_L from the preceding span, it cannot be solved for a single span. And since those values will usually depend, in turn, on values farther upstream, it can, in that case, only be solved as part of a set of simultaneous equations that include the starting point where both y_L and ϕ_L are known. Furthermore, when there is an upstream disturbance influencing y_o and ϕ_o, there is no practical way to express it as a transfer function.

Transfer functions

When y_o and ϕ_o are fixed and the web is being influenced only by inputs z_L, γ_o or γ_L , single span solutions are possible and transfer functions can be written. Guide rollers are in this category. In a steering guide, roller angle γ_L is controlled by the same mechanism that controls lateral movement. So, $\gamma_L = z_L/x_1$, where x_1 is the radius of the pivoting motion. In a displacement guide, γ_o is also controlled by z_L. So, $\gamma_o = z_L/x_1$, where x_1 is the distance between the rollers, L.

Displacement guide confirms validity

For a displacement guide:

$$\phi_o = \gamma_0 = \gamma_L \qquad \gamma_L = \frac{z_L}{L} \qquad y_0 = 0$$

And transfer function is:

$$y_L(s) = z_L \frac{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 + \frac{g_2}{h_2} - \frac{g_2 h_3}{h_2}\right)}{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right)}$$

It can be shown that $g_2 + g_3 = g_1$ and $h_3 + h_2 - 1 = h_1$. Therefore,

$$g_1 + \frac{g_2}{h_2} - \frac{g_2 h_3}{h_2} = g_1 - \frac{g_2 h_1}{h_2}$$

Transfer function is unity (as it should be).

$$y_L = z_L$$

Remote pivot steering guide confirms validity

For a steering guide: $\phi_o = \gamma_0 = 0 \qquad \gamma_L = \dfrac{z_L}{x_1} \qquad y_0 = 0$

And transfer function is:

$$y_L(s) = z_L \frac{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\frac{g_2}{h_2}\frac{1}{x_1}}{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right)}$$

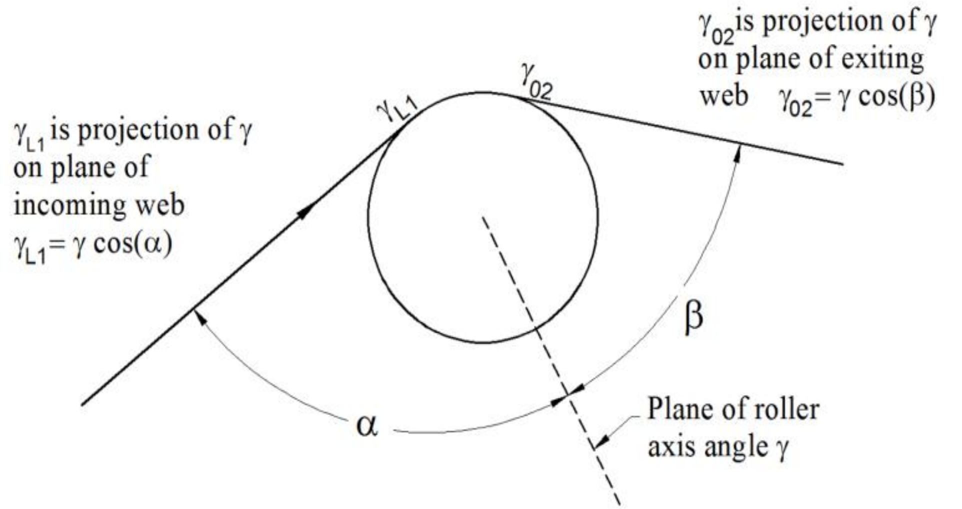
It can be shown that, $y_L = z_L$ when $x_1 = LK_c$

where $$K_c = \frac{g_2}{g_1 h_2 - g_2 h_1} = \frac{1}{KL}\frac{\sinh(KL) - KLa\cosh(KL)}{1 - a\cosh(KL)}$$

K_c is identical to the relationship derived by Shelton for a static web with shear (as it should be).

Effect of roller angle on ϕ_0

If a roller is misaligned, the effective angle will change as the web crosses over it. So, the value of γ_L in ϕ_0 must change.

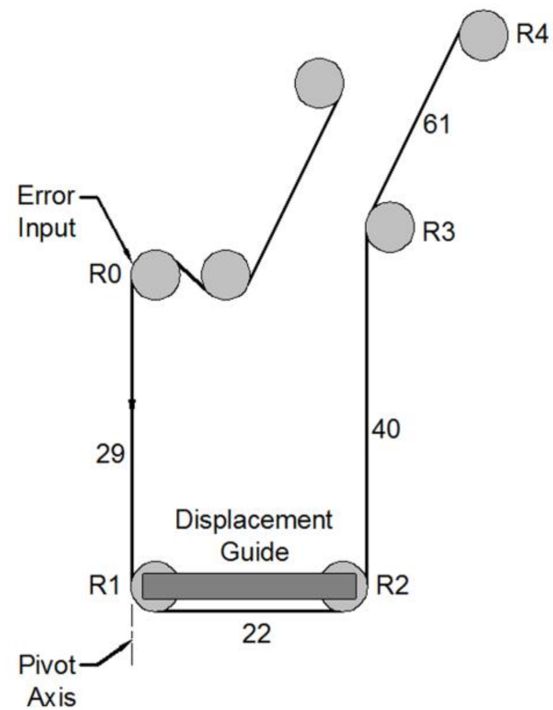


$$\phi_{02} = \phi_{L1} - \gamma_{L1} + \gamma_{02} = \phi_{L1} - \gamma\cos(\alpha) + \gamma\cos(\beta)$$

Sievers’ model for pivoting rollers

The only pivoting rollers in Sievers experiments were part of a displacement guide. So, she applied a rotational coordinate transformation that enabled her to treat them as fixed rollers.

This doesn’t work for single misaligned rollers.



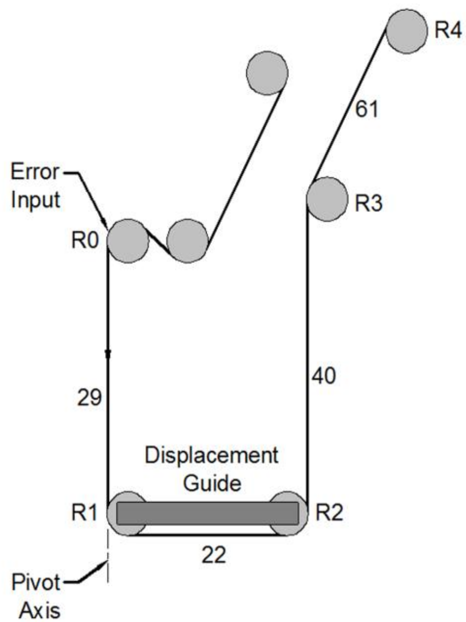
Sievers' displacement guide equation

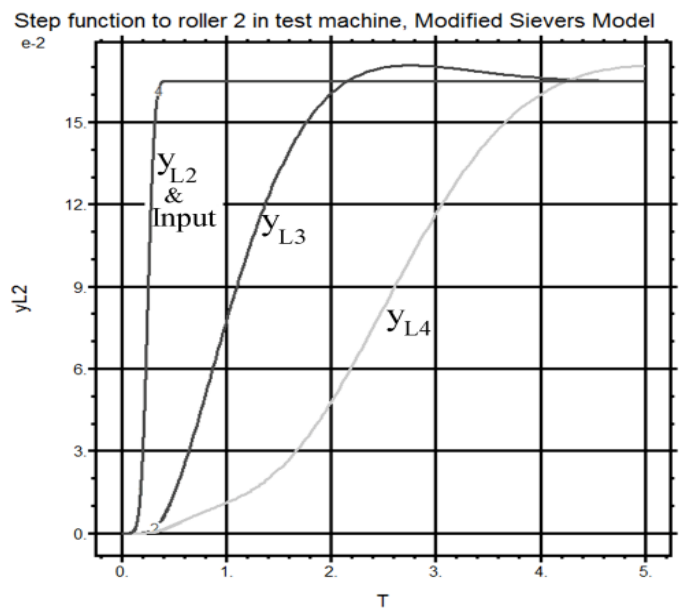
$$\frac{\partial^2 y_L}{\partial t^2} = \left(y_0 - y_L + z_L\right)\frac{v_o{}^2}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\left[\frac{v_o}{L}\left(\frac{\partial z_L}{\partial t} - \frac{\partial y_L}{\partial t}\right)\right]$$
$$+ \frac{v_o{}^2 \phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right) + \frac{\partial^2 z_L}{\partial t^2}$$

This equation was used at roller 2 location. The only difference between it and the other equation is that γ_L isn't present and the roller translation term z_L is added. Simulations using this equation produced results identical to those of the equation described earlier.

Numerical simulation

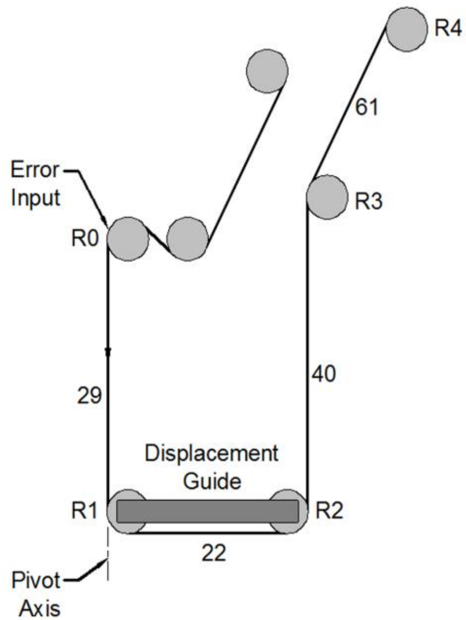
The following simulations were produced with four simultaneous ODEs running on FlexPDE. The model run time was typically 4 to 5 minutes.

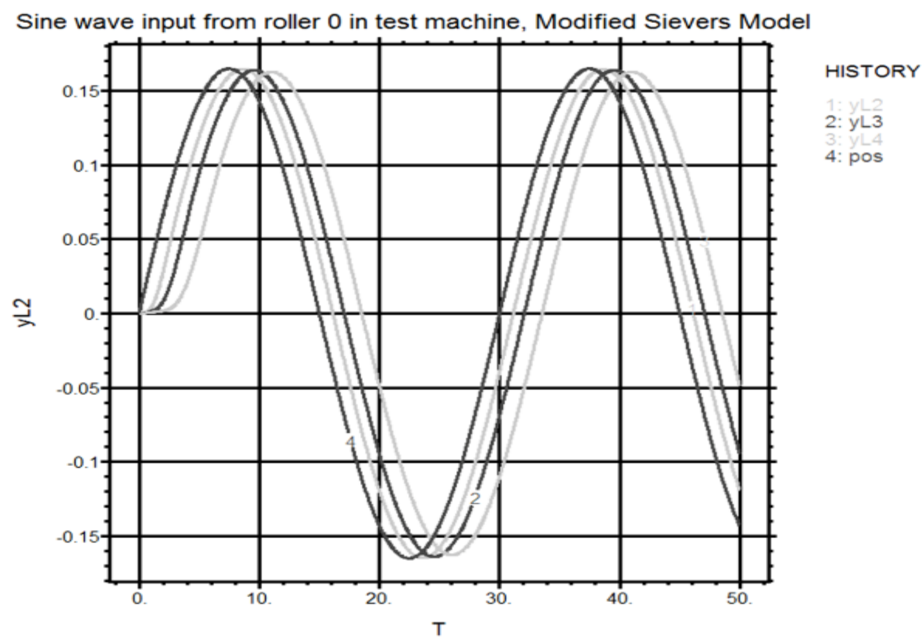




Step function applied to 2nd roller of displacement guide. (Open loop)

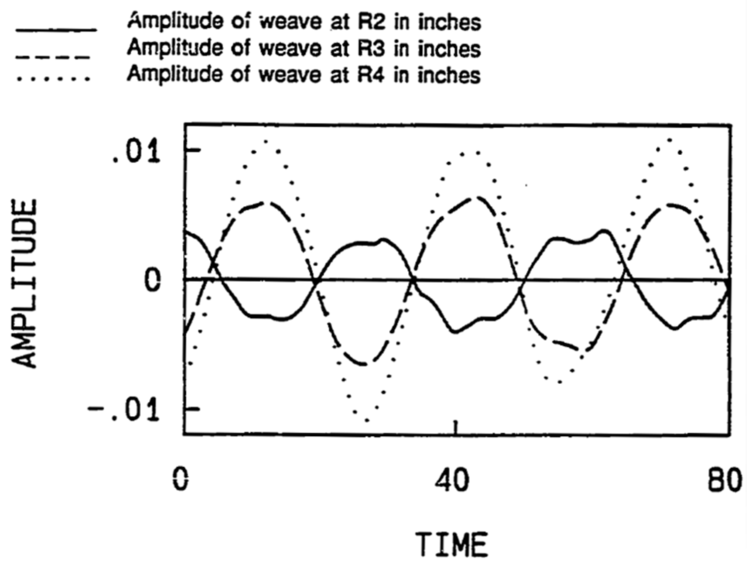
Numerical simulation (Cont.)



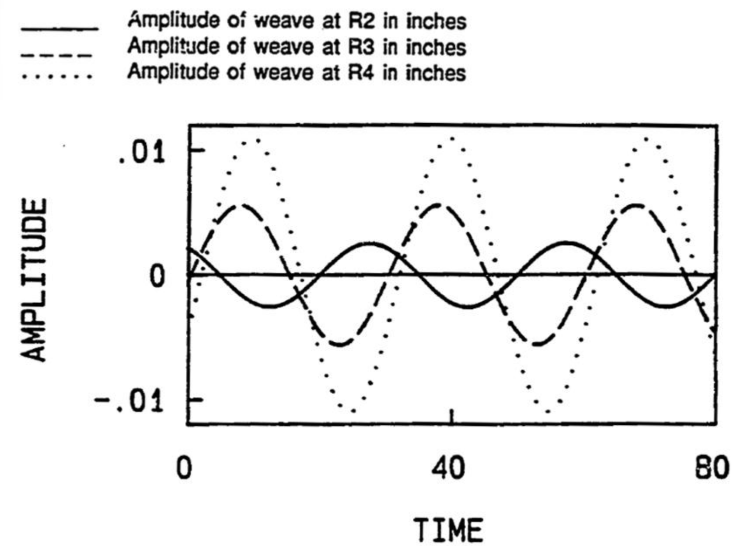


Sine wave disturbance at R0, 2 cycles/min

Numerical simulation (Cont.)



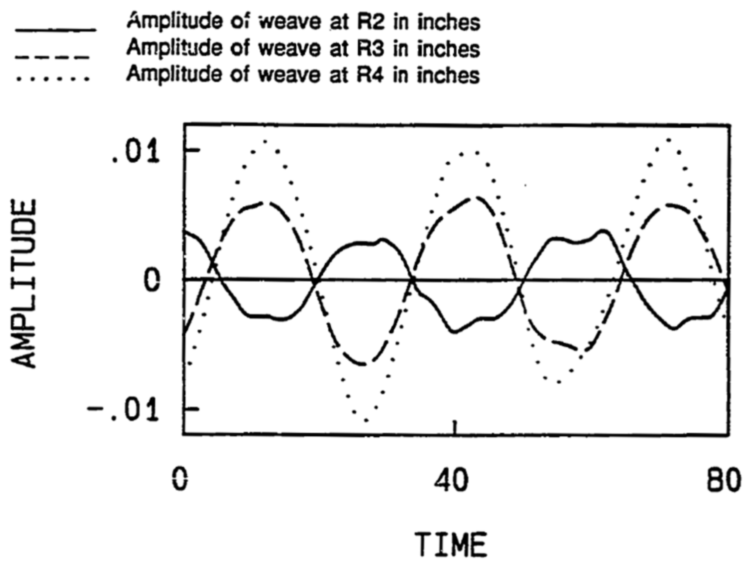
Experimental data from thesis



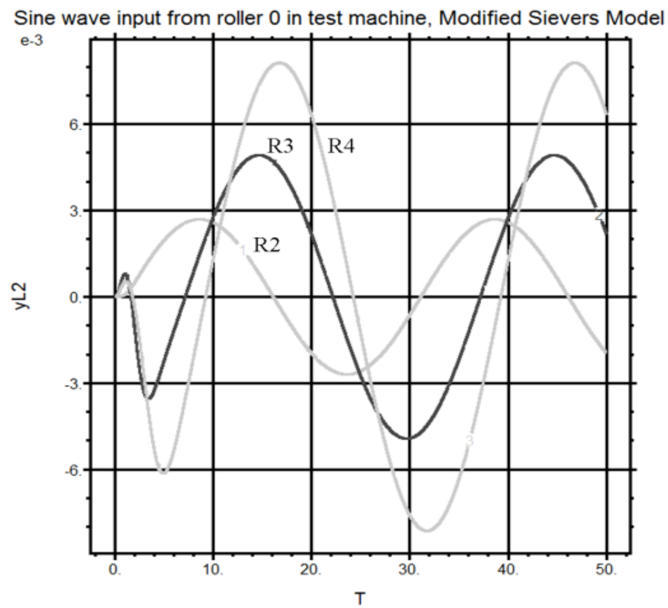
Sievers thesis simulation

A 0.033 Hz weave with an amplitude of 0.165 inch was introduced at roller R0. A guiding system consisting of rollers R1 and R2 with a sensor at R2 reduced the weave. Weave reappeared at R3 and R4 with progressively larger amplitudes.

Numerical simulation (Cont.)



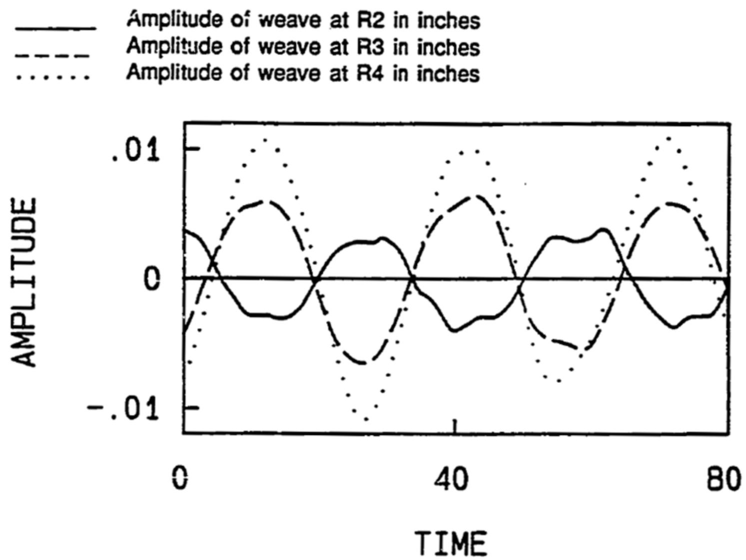
Experimental data from thesis



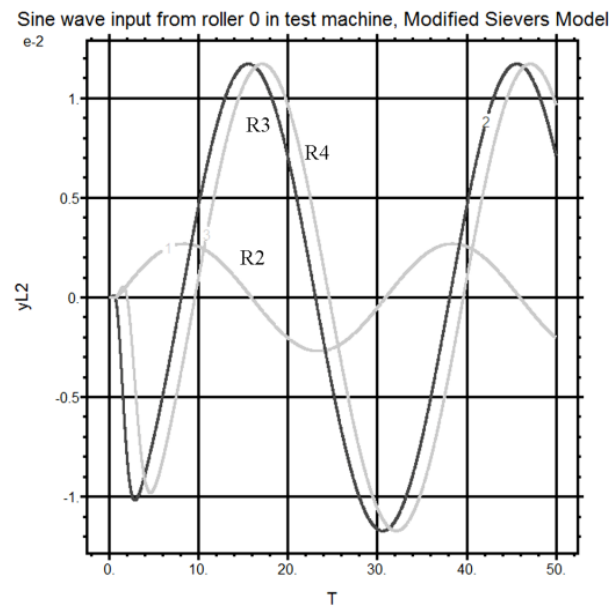
Modified Timoshenko model simulation

Same experiment as last slide. Comparison with modified model simulation.

Numerical simulation (Cont.)



Experimental data from thesis



Modified Euler Bernoulli simulation

Same parameters as previous slide, except comparison with Euler Bernoulli beam. The simulation looks exactly like the one in the thesis. Same amplitudes and phase relations.

Conclusions

- The normal entry rule can be derived by application of the material derivative.
- Lateral position errors can regenerate downstream of a guiding system because variations in slope or cross section angle at the point of control are not eliminated by simple position control systems.
- Both beam models exhibit weave regeneration. Data from four different experiments, each with different operating parameters, show better qualitative agreement using the Timoshenko model simulations than with the Euler Bernoulli model.

Conclusions (Cont.)

- Sievers' model can be improved to eliminate special treatment for pivoting rollers (rotating coordinate system)
- The curvature factor for the modified Sievers model is the same as the value Shelton derived for his static, single span Timoshenko model.
- The lateral acceleration equation should not be viewed in the same light as the normal entry rule.